Exam #2 Solutions

Math 321-A

Friday, October 26, 2001

1. Let $f(x,y) = x^3y+xy^3$. Then ∇f is orthogonal to the level curve $f(x,y) = -2$, and we can find a unit vector by dividing out the gradient's length.

$$\begin{aligned}
f(x,y) &= x^3y + xy^3 \\
\nabla f(x,y) &= \left\langle 3x^2y + y^3, x^3 + 3xy^2 \right\rangle \\
\nabla f(1,-1) &= < -3-1, 1+3 > \\
&= < -4, 4 > \\
\frac{\nabla f(1,-1)}{\|\nabla f(1,-1)\|} &= \frac{< -4, 4 >}{\sqrt{32}} \\
&= \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle
\end{aligned}$$

2.

$$\begin{aligned}
z &= x\sin y + y\cos x \\
dz &= (\sin y - y\sin x)dx + (x\cos y + \cos x)dy
\end{aligned}$$

3. Cylindrical coordinates are appropriate for problems that depend on an axis and the distance between points and that axis.

4.

$$\begin{aligned}
\vec{r}(t) &= \left\langle \cos t^3, \sin t^3 \right\rangle \\
\vec{v}(t) &= \left\langle -3t^2\sin t^3, 3t^2\cos t^3 \right\rangle \\
\frac{ds}{dt} &= \|\vec{v}(t)\| \\
&= 3t^2
\end{aligned}$$

$$\begin{aligned}
\vec{T}(t) &= \frac{\vec{v}(t)}{\|\vec{v}(t)\|} \\
&= \left\langle -\sin t^3, \cos t^3 \right\rangle \\
\frac{d\vec{T}}{dt} &= \left\langle -3t^2 \cos t^3, -3t^2 \sin t^3 \right\rangle \\
\frac{d\vec{T}}{ds} &= \frac{d\vec{T}/dt}{ds/dt} \\
&= \left\langle -\cos t^3, -\sin t^3 \right\rangle \\
\kappa(t) &= \left\| \frac{d\vec{T}}{ds} \right\| \\
&= 1 \\
\vec{N}(t) &= \frac{d\vec{T}/ds}{\left\| d\vec{T}/ds \right\|} \\
&= \left\langle -\cos t^3, -\sin t^3 \right\rangle
\end{aligned}$$

5. The unit tangent vector points in the direction of the curve. The unit normal vector is perpendicular to the unit tangent vector and points in the direction the curve is bending. The unit binormal vector is perpendicular to both the unit tangent and unit normal vectors, and defines (as a vector normal to the plane) the plane of the curve at a given point.

6. The direction of greatest increase is the direction of the gradient. We will find the gradient at the given point and divide out by its length.

$$\begin{aligned}
f(x,y) &= xe^{xy} \\
\nabla f(x,y) &= \left\langle e^{xy} + xye^{xy}, x^2 e^{xy} \right\rangle \\
\nabla f(2,3) &= \left\langle 7e^6, 4e^6 \right\rangle \\
\|\nabla f(2,3)\| &= e^6 \sqrt{65} \\
\frac{\nabla f(2,3)}{\|\nabla f(2,3)\|} &= \left\langle \frac{7}{\sqrt{65}}, \frac{4}{\sqrt{65}} \right\rangle
\end{aligned}$$

7.

$$\begin{aligned}
f(x,y) &= \sin\left(x^2y^3\right) + \cos\left(x^3y^2\right) \\
&= 1 \\
\frac{dy}{dx} &= -\frac{\partial f/\partial x}{\partial f/\partial y}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2xy^3\cos\left(x^2y^3\right) - 3x^2y^2\sin\left(x^3y^2\right)}{3x^2y^2\cos\left(x^2y^3\right) - 2x^3y\sin\left(x^3y^2\right)} \\
&= -\frac{2y^2\cos\left(x^2y^3\right) - 3xy\sin\left(x^3y^2\right)}{3xy\cos\left(x^2y^3\right) - 2x^2\sin\left(x^3y^2\right)}
\end{aligned}
$$

8. The surface area is found by integrating the circumference (2π times the radius measured from the axis, i.e., $2\pi x$) times the differential of arc-length ($ds = \sqrt{1 + (dy/dx)^2}\,dx$).

$$
\begin{aligned}
S &= \int_0^2 2\pi(\text{radius})\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_0^2 2\pi x\sqrt{1 + (2 - 2x)^2}\,dx \\
&= 18.5849
\end{aligned}
$$