Final Exam Solutions

Math 321-A

Monday, December 10, 2001

1.

$$\begin{aligned}
c &= \sqrt{a^2+b^2} \\
&= \sqrt{9+4} \\
&= \sqrt{13} \\
e &= \frac{c}{a} \\
&= \frac{\sqrt{13}}{3} \\
&= 1.202
\end{aligned}$$

2.

$$\begin{aligned}
r &= 2\cos\theta \\
x &= r\cos\theta \\
y &= r\sin\theta \\
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\
&= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\
&= \frac{(-2\sin\theta)\sin\theta + (2\cos\theta)\cos\theta}{(-2\sin\theta)\cos\theta - (2\cos\theta)\sin\theta} \\
&= \frac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta} \\
&= -\frac{\cos(2\theta)}{\sin(2\theta)} \\
&= -\cot(2\theta) \\
\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} &= -\cot\left(\frac{2\pi}{3}\right)
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{3}} \\ &= 0.577 \end{aligned}$$

3.

$$\begin{aligned} \vec{r}(t) &= \left\langle t^3 - 3t, 3t^2 \right\rangle \\ \vec{v}(t) &= \left\langle 3t^2 - 3, 6t \right\rangle \\ \|\vec{v}(t)\| &= \sqrt{\left(3t^2 - 3\right)^2 + (6t)^2} \\ &= \sqrt{9t^4 + 18t^2 + 9} \\ &= 3t^2 + 3 \\ \text{arc-length} &= \int_a^b \|\vec{v}(t)\| \, dt \\ &= \int_{-1}^{1} \left(3t^2 + 3\right) \, dt \\ &= t^3 + 3t\Big|_{-1}^{1} \\ &= (1+3) - (-1-3) \\ &= 8 \end{aligned}$$

4.

$$\begin{aligned} f(x,y,z) &= x + y^2 + z^3 \\ \nabla f(x,y,z) &= \left\langle 1, 2y, 3z^2 \right\rangle \\ \|\nabla f(x,y,z)\| &= \sqrt{1^2 + (2y)^2 + \left(3z^2\right)^2} \\ &= \sqrt{1 + 4y^2 + 9z^4} \\ \frac{\nabla f}{\|\nabla f\|} &= \frac{\left\langle 1, 2y, 3z^2 \right\rangle}{\sqrt{1 + 4y^2 + 9z^4}} \end{aligned}$$

5. There are two supplementary angles, one acute and the other obtuse. We will check our answer to see if it is obtuse; if it is, we will find its supplement.

   The angle between the planes is the same as the angle between the two normal vectors $\vec{n}_1 =< 3, -4, 5 >$ and $\vec{n}_2 =< 1, -2, -3 >$.

$$\begin{aligned}
\cos\theta &= \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \, \|\vec{n}_2\|} \\
&= \frac{3\cdot 1 + (-4)\cdot(-2) + 5\cdot(-3)}{\sqrt{3^2+(-4)^2+5^2}\sqrt{1^2+(-2)^2+(-3)^2}} \\
&= -\frac{2}{5\sqrt{7}} \\
\theta &= \cos^{-1}\left(-\frac{2}{5\sqrt{7}}\right) \\
&= 1.723 \\
&> \frac{\pi}{2} \\
\pi - \theta &= 1.419
\end{aligned}$$

6. We will set the components of the lines equal and try to solve for s and t.

$$\begin{aligned}
\vec{r}_1(s) &= < 3s+1, -2s+5, s > \\
&= \vec{r}_2(t) \\
&= < 2t-1, -t, -t > \\
z \text{ component: } s &= -t \\
y \text{ component: } -2s+5 &= 2t+5 \\
&= -t \\
t &= -\frac{5}{3} \\
s &= \frac{5}{3} \\
x \text{ component: } 3s+1 &= 2t-1 \\
3\left(\frac{5}{3}\right)+1 &= 2\left(-\frac{5}{3}\right)-1 \\
6 &= -\frac{13}{3}
\end{aligned}$$

Since we get a contradiction, the lines do not intersect.

7. Let $P = (1,3,5)$. Let Q be any point on the plane, e.g., $(4,0,0)$. $Q - P = (3,-3,-5)$ is a vector from P to the plane. $\vec{n} =< 1,1,1 >$ is a normal vector for the plane. The projection of $Q - P$ along $\vec{n}$ is a perpendicular vector from P to the plane and its length is the perpendicular distance.

$$\operatorname{proj}_{\vec{n}} Q - P = \frac{(Q-P)\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$$

$$\begin{aligned}
&= \frac{3\cdot 1 + (-3)\cdot 1 + (-5)\cdot 1}{1^2+1^2+1^2} < 1,1,1 > \\
&= -\frac{5}{3} < 1,1,1 > \\
\|\operatorname{proj}_{\vec{n}} Q - P\| &= \frac{5}{3}\sqrt{1^2+1^2+1^2} \\
&= \frac{5}{\sqrt{3}} \\
&= 2.887
\end{aligned}$$

8.

$$\begin{aligned}
\vec{r}(t) &= < t, \sin t, \cos t > \\
\vec{v}(t) &= < 1, \cos t, -\sin t > \\
\vec{a}(t) &= < 0, -\sin t, -\cos t > \\
\|\vec{v}(t)\| &= \sqrt{1^2 + (\cos t)^2 + (-\sin t)^2} \\
&= \sqrt{2} \\
\vec{T} &= \frac{\vec{v}(t)}{\|\vec{v}(t)\|} \\
&= \frac{1}{\sqrt{2}} < 1, \cos t, -\sin t > \\
\vec{v}\times\vec{a} &= < (\cos t)(-\cos t) - (\cos t)(0), (-\sin t)(0) - (1)(-\cos t), \\
&\quad (1)(-\sin t) - (\cos t)(0) > \\
&= < -1, \cos t, -\sin t > \\
\|\vec{v}\times\vec{a}\| &= \sqrt{(-1)^2 + (\cos t)^2 + (-\sin t)^2} \\
&= \sqrt{2} \\
\vec{B} &= \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|} \\
&= \frac{1}{\sqrt{2}} < -1, \cos t, -\sin t > \\
\vec{N} &= \vec{B}\times\vec{T} \\
&= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} < (\cos t)(-\sin t) - (-\sin t)(\cos t), \\
&\quad (-\sin t)(1) - (-1)(-\sin t), (-1)(\cos t) - (\cos t)(1) > \\
&= < 0, -\sin t, -\cos t >
\end{aligned}$$

9.

$$\begin{aligned} f(x,y) &= x^2y + xy^2 - 12 \\ \frac{dy}{dx} &= -\frac{f_x}{f_y} \\ &= -\frac{2xy+y^2}{x^2+2xy} \end{aligned}$$

10.

$$\begin{aligned} f(x,y) &= ye^x + 3e^x - y \\ f_x &= ye^x + 3e^x \\ &= (y+3)e^x \\ f_y &= e^x - 1 \\ \text{Critical point: } x &= 0 \\ y &= -3 \\ f_{xx} &= ye^x + 3e^x \\ f_{xy} &= e^x \\ f_{yy} &= 0 \\ \text{At } (0,-3)\text{: } D &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= 0 \cdot 0 - 1^2 \\ &= -1 \\ &< 0 \end{aligned}$$

and the point $(0,-3)$ is a saddle point. $f(0,-3) = (-3) \cdot e^0 + 3 \cdot e^0 - (-3) = 3$.

11. We need to integrate $\|\vec{r}_u \times \vec{r}_v\|$ over the given region.

$$\begin{aligned} \vec{r}(u,v) &= < u+v, u-v, v-u > \\ \vec{r}_u &= < 1, 1, -1 > \\ \vec{r}_v &= < 1, -1, 1 > \\ \vec{r}_u \times \vec{r}_v &= < 1 \cdot 1 - (-1) \cdot (-1), (-1) \cdot 1 - 1 \cdot 1, \\ & \quad 1 \cdot (-1) - 1 \cdot 1 > \\ &= < 0, -2, -2 > \\ \|\vec{r}_u \times \vec{r}_v\| &= \sqrt{0^2 + (-2)^2 + (-2)^2} \end{aligned}$$

$$\begin{aligned}
&= 2\sqrt{2} \\
\text{Surface area} &= \int_2^3 \int_0^1 2\sqrt{2}\, du\, dv \\
&= \int_2^3 2\sqrt{2}\, dv \\
&= 2\sqrt{2} \\
&= 2.828
\end{aligned}$$

12.

$$\begin{aligned}
\vec{F}(x,y) &= \nabla f(x,y) \\
&= \left\langle 2xe^y, x^2e^y + 3y^2 \right\rangle \\
&= \langle f_x, f_y \rangle \\
f_x &= 2xe^y \\
f(x,y) &= x^2e^y + C_1(y) \\
f_y &= x^2e^y + C_1'(y) \quad \text{from previous line} \\
&= x^2e^y + 3y^2 \quad \text{from vector field} \\
C_1'(y) &= 3y^2 \\
C_1(y) &= y^3 + C_2 \\
f(x,y) &= x^2e^y + y^3 + C_2
\end{aligned}$$

13.

$$\begin{aligned}
\vec{F}(x,y,z) &= \left\langle x+y+z, x^2, y^2 \right\rangle \\
\operatorname{curl} \vec{F} &= \left\langle \frac{\partial y^2}{\partial y} - \frac{\partial x^2}{\partial z}, \frac{\partial x+y+z}{\partial z} - \frac{\partial y^2}{\partial x}, \frac{\partial x^2}{\partial x} - \frac{\partial x+y+z}{\partial y} \right\rangle \\
&= \langle 2y, 1, 2x-1 \rangle
\end{aligned}$$

14. All of the conic sections have the same form of equation in polar coordinates, while their equations are very different in Cartesian coordinates.

15.

$$\vec{r}(t) = \left\langle t^2, t^3, t^4 \right\rangle$$

$$\begin{aligned}
\vec{v}(t) &= \left\langle 2t, 3t^2, 4t^3 \right\rangle \\
\vec{v}(1) &= \langle 2, 3, 4 \rangle \\
\vec{a}(t) &= \left\langle 2, 6t, 12t^2 \right\rangle \\
\vec{a}(1) &= \langle 2, 6, 12 \rangle \\
\vec{v}(1) \times \vec{a}(1) &= \langle 3 \cdot 12 - 4 \cdot 6, 4 \cdot 2 - 2 \cdot 12, 2 \cdot 6 - 3 \cdot 2 \rangle \\
&= < 12, -16, 6 > \\
\|\vec{v}(1)\| &= \sqrt{2^2 + 3^2 + 4^2} \\
&= \sqrt{29} \\
\|\vec{v}(1) \times \vec{a}(1)\| &= \sqrt{12^2 + (-16)^2 + 6^2} \\
&= 2\sqrt{109} \\
\kappa(1) &= \frac{\|\vec{v}(1) \times \vec{a}(1)\|}{\|\vec{v}(1)\|^3} \\
&= \frac{2\sqrt{109}}{29\sqrt{29}} \\
&= 0.1337
\end{aligned}$$

16. The region is described by $0 \leq y \leq \sqrt{x}$ and $0 \leq x \leq 4$.

$$\begin{aligned}
\text{Area} &= \int_0^4 \int_0^{\sqrt{x}} 1 \, dy \, dx \\
&= \int_0^4 y|_{y=0}^{y=\sqrt{x}} \, dx \\
&= \int_0^4 \sqrt{x} \, dx \\
&= \left. \frac{x^{3/2}}{3/2} \right|_0^4 \\
&= \frac{16}{3} \\
M_y &= \int_0^4 \int_0^{\sqrt{x}} x \, dy \, dx \\
&= \int_0^4 xy|_{y=0}^{y=\sqrt{x}} \, dx \\
&= \int_0^4 x^{3/2} \, dx \\
&= \left. \frac{x^{5/2}}{5/2} \right|_0^4
\end{aligned}$$

$$
\begin{aligned}
&= \frac{64}{5} \\
M_x &= \int_0^4 \int_0^{\sqrt{x}} y\,dy\,dx \\
&= \int_0^4 \left.\frac{y^2}{2}\right|_0^{\sqrt{x}} \,dx \\
&= \int_0^4 \frac{x}{2}\,dx \\
&= \left.\frac{x^2}{4}\right|_0^4 \\
&= 4 \\
\bar{x} &= \frac{M_y}{\text{Area}} \\
&= \frac{64/5}{16/3} \\
&= \frac{12}{5} \\
&= 2.4 \\
\bar{y} &= \frac{M_x}{\text{Area}} \\
&= \frac{4}{16/3} \\
&= \frac{3}{4} \\
&= 0.75
\end{aligned}
$$

17.

$$
\begin{aligned}
f(x,y) &= 2x+3y \\
\vec{r}(t) &= <\cos t, \sin t> \\
f(\vec{r}(t)) &= 2\cos t + 3\sin t \\
\vec{v}(t) &= <-\sin t, \cos t> \\
\|\vec{v}(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2} \\
&= 1 \\
\int_C f\,ds &= \int_a^b f(\vec{r}(t))\,\|\vec{v}(t)\|\,dt \\
&= \int_0^\pi (2\cos t + 3\sin t)(1)\,dt \\
&= 2\sin t - 3\cos t|_0^\pi
\end{aligned}
$$

$$\begin{aligned}
&= (0-(-3))-(0-3) \\
&= 6
\end{aligned}$$

18. The region has $0 \leq \rho \leq 1$ going from the origin out to the boundary hemisphere. The region has θ taking on all values from 0 to 2π as it sweeps around the xy-plane. The region has ϕ only going from 0 (the positive z-axis) down to $\pi/2$ (the base in the xy-plane).

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho)\left(\rho^2\sin\phi\right)\,d\rho\,d\phi\,d\theta &= \int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^4}{4}\sin\phi\Big|_{\rho=0}^{\rho=1}\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2} \frac{\sin\phi}{4}\,d\phi\,d\theta \\
&= \int_0^{2\pi} -\frac{\cos\phi}{4}\Big|_{\phi=0}^{\phi=\pi/2}\,d\theta \\
&= \int_0^{2\pi} \left(0-\left(-\frac{1}{4}\right)\right)d\theta \\
&= \int_0^{2\pi} \frac{1}{4}\,d\theta \\
&= \frac{\pi}{2} \\
&= 1.5708
\end{aligned}$$

19.

$$\begin{aligned}
\vec{r}(t) &= \left\langle e^t, e^{-t}, 0\right\rangle \\
\vec{v}(t) &= \left\langle e^t, -e^{-t}, 0\right\rangle \\
\|\vec{v}\| &= \sqrt{e^{2t}+e^{-2t}} \\
\vec{a}(t) &= \left\langle e^t, e^{-t}, 0\right\rangle \\
\vec{v}\times\vec{a} &= \left\langle 0-0, 0-0, e^t\cdot e^{-t}-\left(-e^{-t}\right)\cdot e^t\right\rangle \\
&= <0,0,2> \\
\|\vec{v}\times\vec{a}\| &= 2 \\
a_{\vec{N}} &= \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|} \\
&= \frac{2}{\sqrt{e^{2t}+e^{-2t}}} \\
a_{\vec{T}} &= \frac{\vec{v}\cdot\vec{a}}{\|\vec{v}\|}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^t \cdot e^t + (-e^{-t}) \cdot e^{-t}}{\sqrt{e^{2t} + e^{-2t}}} \\
&= \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}
\end{aligned}
$$

20. The directional derivative is the dot product of the gradient and the direction vector.

$$
\begin{aligned}
f(x,y) &= x^2 - y^2 \\
\nabla f(x,y) &= < 2x, -2y > \\
D_{\vec{u}} f(x,y) &= \nabla f \cdot \vec{u} \\
&= 2x \cdot \frac{\sqrt{3}}{2} + (-2y) \cdot \left(-\frac{1}{2}\right) \\
&= \sqrt{3}x + y
\end{aligned}
$$