

Final Exam Solutions

Math 121-H

Saturday, December 6, 2003

1. This is an exponential model.

$$\begin{aligned}f(n) &= -5f(n-1) \\ &= (-5)(-5)f(n-2) \\ &= \dots \\ &= (-5)^n f(0) \\ &= 7(-5)^n\end{aligned}$$

2. The point A can only be left fixed or sent to D . Similarly the point B can only be left fixed or sent to C .

If A and B are left fixed, we have a a rotation by 0° , i.e., the identity.

If A is left fixed, and B and C are switched, we have b a vertical reflection.

If A and D are switched but B is left fixed, we have c a horizontal reflection.

If A and D are switched, and B and C are switched, we have d a rotation by 180° .

Each of these symmetries is its own inverse. The composition of two rotations or two reflections is a rotation, and the composition of a rotation and reflection (in either order) is a reflection.

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

3. The easiest example is a rotation by $360^\circ/5 = 72^\circ$.
4. We need to find the second derivative of $f(x)$.

$$\begin{aligned}
f(x) &= \frac{x^3}{x+2} \\
f'(x) &= \frac{(x+2)(x^3)' - x^3(x+2)'}{(x+2)^2} \\
&= \frac{(x+2)(3x^2) - x^3(1)}{(x+2)^2} \\
&= \frac{3x^3 + 6x^2 - x^3}{(x+2)^2} \\
&= \frac{2x^3 + 6x^2}{(x+2)^2} \\
f''(x) &= \frac{(x+2)^2(2x^3 + 6x^2)' - (2x^3 + 6x^2)((x+2)^2)'}{((x+2)^2)^2} \\
&= \frac{(x+2)^2(6x^2 + 12x) - (2x^3 + 6x^2)(2(x+2)(x+2)')}{(x+2)^4} \\
&= \frac{(x+2)^2(6x^2 + 12x) - (2x^3 + 6x^2)(2(x+2)(1))}{(x+2)^4} \\
&= \frac{(x+2)(6x^2 + 12x) - (2x^3 + 6x^2)(2)}{(x+2)^3} \\
&= \frac{6x^3 + 12x^2 + 12x^2 + 24x - 4x^3 - 12x^2}{(x+2)^3} \\
&= \frac{2x^3 + 12x^2 + 24x}{(x+2)^3} \\
&= \frac{2x(x^2 + 6x + 12)}{(x+2)^3}
\end{aligned}$$

Note that $x^2 + 6x + 12$ has no real solutions since $b^2 - 4ac = 36 - 48 = -12 < 0$.

The hypercritical numbers are 0 ($f''(x) = 0$) and -2 ($f''(x)$ undefined). The only hypercritical point is $(0, 0)$ since $f(-2)$ is undefined.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
Test Value	-3	-1	1
Sign of $f''(x)$	$+$	$-$	$+$
Concavity	up	down	up

The curve is concave up on $(-\infty, -2) \cup (0, \infty)$ and concave down on $(-2, 0)$. There is a single inflection point at $(0, 0)$.

5. We first integrate, then solve for the constant of integration.

For the integration, let $u = 3x$, $du = 3 dx$, $dx = du/3$.

$$\begin{aligned}f'(x) &= e^{3x} \\f(x) &= \int e^{3x} dx \\&= \int e^u \frac{du}{3} \\&= \frac{e^u}{3} + C \\&= \frac{e^{3x}}{3} + C \\f(2) &= 5 \\&= \frac{e^6}{3} + C \\C &= 5 - \frac{e^6}{3} \\f(x) &= \frac{e^{3x}}{3} + 5 - \frac{e^6}{3}\end{aligned}$$

6. We start by finding the first derivative.

$$\begin{aligned}f(x) &= x + 4x^{-2} \\f'(x) &= 1 - 8x^{-3} \\&= 1 - \frac{8}{x^3} \\&= \frac{x^3 - 8}{x^3}\end{aligned}$$

There are two critical numbers: $x = 2$ makes the numerator 0, and $x = 0$ makes the denominator 0. There is only one critical point at $(2, 3)$.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Test Value	-1	1	3
Sign of $f'(x)$	+	-	+
Conclusion	increasing	decreasing	increasing

The curve is increasing on $(-\infty, 0) \cup (2, \infty)$ and decreasing on $(0, 2)$. There is a relative minimum at $(2, 3)$.

7. First differences are analogous to first derivatives: their sign indicated whether or not the data were increasing or decreasing. Second differences are analogous to second derivatives: their sign indicated whether or not the data was concave up or down.

8. x' is the slope of the line $y = x$, namely, 1.

Or:

$$\begin{aligned}
 f(x) &= x \\
 f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} 1 \\
 &= 1
 \end{aligned}$$

9.

n	$f(n)$
1	10
2	$\frac{10}{2} = 5$
3	$\frac{10}{2} = 5$
4	$\frac{5}{3}$
5	$\frac{5/3}{4} = \frac{5}{12}$
6	$\frac{5/12}{5} = \frac{1}{12}$

10. We will use the substitution $u = x^3 + 5$. $du = 3x^2 dx$, $x^2 dx = du/3$.
 When $x = 0$, $u = 0^3 + 5 = 5$, and when $x = 5$, $u = 5^3 + 5 = 130$.

$$\begin{aligned}\int_0^5 \frac{x^2 dx}{x^3 + 5} &= \int_5^{130} \frac{du/3}{u} \\ &= \frac{1}{3} \int_5^{130} \frac{du}{u} \\ &= \frac{1}{3} \ln |u| \Big|_5^{130} \\ &= \frac{1}{3} (\ln 130 - \ln 5) \\ &= \frac{\ln 26}{3}\end{aligned}$$