Exam #3 Solutions

Math 425-A

Monday, November 19, 2007

1. Since $e^{nx^2} \geq e^0 = 1$,
$$\frac{1}{n^2 e^{nx^2}} \leq \frac{1}{n^2}$$
Since
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$
converges, by the Weierstrass M-test, the original series of functions converges uniformly.

2. All of the Cauchy criteria refer to things (constants or functions) getting closer to each other. More formally, given an $\epsilon > 0$ we find an N such that $n > m \geq N$ implies that the things are less than ϵ apart. The criteria then imply convergence of some kind.

3. In continuity, we require that for any given $x = a$ given an $\epsilon > 0$ we can find a $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \epsilon$. The δ can depend on a.

   For uniform continuity, we want to find such a δ that works for all such a simultaneously. That is usually impossible, so uniform continuity is more restrictive than continuity.

4.
   (a) The identify function $f(x) = x$ from $(0, 1)$ to $(0, 1)$
   (b) The function $f(x) = x^2$ from $(-3, -2) \cup (2, 3)$ to $(4, 9)$
   (c) This is impossible since a continuous function sends connected sets to connected sets; an interval is connected but the union of two separated intervals is not.
   (d) The identify function $f(x) = x$ from $(-3, -2) \cup (2, 3)$ to $(-3, -2) \cup (2, 3)$

5. We will use the ratio test.
$$\lim_{n\to\infty} \left| \frac{x^{n+1}/3^{2n+3}}{x^n/3^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x}{9} \right|$$
$$= \frac{|x|}{9}$$
This will be less than 1 when $|x| < 9$, and the radius of convergence is 9.

6. Since $f_n(x) = \cos^n(2\pi n! x)$ is a continuous function of x for each $n \in \mathbb{N}$, if the convergence were uniform, the limit would be continuous.

   The limit function is discontinuous at every x. Therefore the convergence can not be uniform.

7.
$$f(x) = \frac{3x^2 + 2x + 4}{x^2 - 2x - 3}$$
$$f'(x) = \frac{\left(x^2 - 2x - 3\right)\left(3x^2 + 2x + 4\right)' - \left(3x^2 + 2x + 4\right)\left(x^2 - 2x - 3\right)'}{\left(x^2 - 2x - 3\right)^2}$$

Quotient Rule

$$= \frac{\left(x^2 - 2x - 3\right)\left(\left(3x^2\right)' + (2x)' + (4)'\right) - \left(3x^2 + 2x + 4\right)\left(\left(x^2\right)' - (2x)' - (3)'\right)}{(x^2 - 2x - 3)^2}$$

Sum and Difference Rules

$$= \frac{\left(x^2 - 2x - 3\right)\left(3\left(x^2\right)' + 2\,(x)' + 0\right) - \left(3x^2 + 2x + 4\right)((2x) - 2\,(x)' - 0)}{(x^2 - 2x - 3)^2}$$

Constant, Constant Multiple, Power Rules

$$= \frac{\left(x^2 - 2x - 3\right)(3\,(2x) + 2\,(1)) - \left(3x^2 + 2x + 4\right)((2x) - 2\,(1) - 0)}{(x^2 - 2x - 3)^2}$$

Identity, Power Rules

$$\begin{aligned}
&= \frac{\left(x^2 - 2x - 3\right)(6x + 2) - \left(3x^2 + 2x + 4\right)(2x - 2)}{(x^2 - 2x - 3)^2} \\
&= \frac{\left(6x^3 + 2x^2 - 12x^2 - 4x - 18x - 6\right) - \left(6x^3 - 6x^2 + 4x^2 - 4x + 8x - 8\right)}{(x^2 - 2x - 3)^2} \\
&= \frac{-8x^2 - 26x + 2}{(x^2 - 2x - 3)^2}
\end{aligned}$$

8. A continuous function sends compact sets to compact sets. $[0, 1]$ is closed and bounded, and compact by the Heine-Borel Theorem. $(0, 1)$ is not closed (0 is a limit point that is not contained by $(0, 1)$), not compact, and there can be no continuous, onto function from $[0, 1]$ to $(0, 1)$.

9. Since $\sin(\pi(2)) = 0 = \ln(2 - 1)$, we will apply l'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 2} \frac{(\sin(\pi x))'}{(\ln(x - 1))'} &= \lim_{x\to 2} \frac{\pi\cos(\pi x)}{1/(x - 1)} \\
&= \frac{\pi\cos(2\pi)}{1/(2 - 1)} \\
&= \pi \\
\lim_{x\to 2} \frac{\sin(\pi x)}{\ln(x - 1)} &= \pi
\end{aligned}$$

10. Since f is bounded, there is some constant M such that $|f(x)| \le M$ for all x, i.e., $-M \le f(x) \le M$.

    Then $-M \le f(2x) \le M$ for all x, and $5 - M \le f(2x) + 5 \le M + 5$. Let $M_2 = \max(|5 - M|, |M + 5|)$. Then $|f(2x) + 5| \le M_2$ for all x.