Final Exam Solutions

Math 425-A

Wednesday, December 12, 2007

1. One of the criterion for continuity at $x = c$ is that

$$\lim_{x \to c} f(x) = f(c)$$

We will use the differentiability at $x = c$ to prove that limit.

$$\begin{aligned}
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= f'(c) \\
\lim_{x \to c} (f(x) - f(c)) &= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right)(x - c) \\
&= f'(c) \cdot 0 \\
&= 0 \\
\lim_{x \to c} f(x) &= \lim_{x \to c} (f(x) - f(c) + f(c)) \\
&= 0 + f(c) \\
&= f(c)
\end{aligned}$$

2. A perfect set is a set that contains all of its limit points and such that every point is a limit point.

We showed that the Cantor set was perfect first by showing that it was closed (and thus contained all of its limit points) since it was the intersection of a collection of closed sets (its approximations).

Then we showed that every point in the Cantor set was a limit point by constructing a sequence of points converging to it. We used the approximations to the Cantor set, to find endpoints on a segment containing the given point, and choosing one of them for our sequence.

3. One possible example is $f(x) = |x|$ on $[-1, 1]$. The slope between the endpoints is $(1 - 1)/(1 - (-1)) = 0$, but there is no place on the graph of $y = |x|$ with slope 0.

4. Let $\epsilon = L/2 > 0$. Then there is an N such that $n \geq N$ implies that $|x_n - L| < L/2$ since $x_n \to L$. When this happens we have $-L/2 < x_n - L < L/2$ and $0 < L/2 < x_n$.

5. We'll use the Ratio Test.

$$\begin{aligned}
\frac{(n+1)!(n+1)!x^{n+1}/(2n+2)!}{n!n!x^n/(2n)!} &= \frac{(n+1)!(n+1)!(2n)!x^{n+1}}{n!n!(2n+2)!x^n} \\
&= \frac{(n+1)^2 x}{(2n+1)(2n+2)} \\
\lim_{n \to \infty} \left| \frac{(n+1)^2 x}{(2n+1)(2n+2)} \right| &= \frac{|x|}{4}
\end{aligned}$$

We have convergence for $|x|/4 < 1$, i.e., for $|x| < 4$, and the radius of convergence is $R = 4$.

6. We used the Extreme Value Theorem to show that the function took an absolute maximum and an absolute minimum. If one of these extrema were on the interior of the interval, we used the Interior Extremum Theorem to show that the slope at that point was 0.

7. A set is disconnected if it can be written as the union of two non-empty separated sets U and V. Let $U = [0, 1]$ and $V = [2, 3]$.

   U and V are disjoint. We need to show that neither contains a limit point of the other.

   Suppose that U did contain a limit point x of V. Then every open interval containing x would contain another point that was in V. But $x \in U \subseteq (-1, 2)$, which does not contain any points of V, a contradiction. Therefore U does not contain any limit points of V. Similarly V does not contain any limit points of U, and U and V are separated.

   Therefore $[0, 1] \cup [2, 3]$ are disconnected.

8.
$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{\left(x^2 - 5x\right) - \left(c^2 - 5c\right)}{x - c} \\ &= \lim_{x \to c} \frac{x^2 - c^2 - 5x + 5c}{x - c} \\ &= \lim_{x \to c} x + c - 5 \\ &= 2c - 5 \end{aligned}$$

9. We will use the Weierstrass M-test.
$$0 \leq \frac{x^2}{3^n\left(x^2 + 1\right)} < \frac{1}{3^n}$$
   and since $\sum 1/3^n$ converges (it is a geometric series with ratio $1/3$), the original series converges uniformly.

10. Every closed interval is compact, and every continuous function with compact domain is uniformly continuous.

    An example from class is $f(x) = 1/x$ on $(0, 1]$. We can show it is not uniformly continuous with the sequential criterion. Let $x_n = 1/(n + 1)$ and $y_n = 1/n$. Since both $x_n \to 0$ and $y_n \to 0$, $x_n - y_n \to 0$. $f(x_n) - f(y_n) = (n + 1) - n = 1$, and consequently we can never make $f(x_n) - f(y_n)$ small.

11. The proof of the Intermediate Value Theorem rests on the fact that continuous maps send connected sets to connected sets.

    The proof of the Extreme Value Theorem rests on the fact that continuous maps send compact sets to compact sets.

12. The domain must also be connected. An example would be:
$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 5 & 2 < x < 3 \end{cases}$$

13.

Interval	Δx_k	m_k	$m_k \Delta x_k$	M_k	$M_k \Delta x_k$
$[0, 0.5]$	0.5	0	0	0.125	0.0625
$[0.5, 1]$	0.5	0.125	0.0625	1	0.5
$[1, 1.5]$	0.5	1	0.5	3.375	1.6875
$[1.5, 2]$	0.5	3.375	1.6875	8	4
	$b - a = 2$		$L(f, P) = 2.25$		$U(f, P) = 6.25$

14. We were showing that for any $\epsilon > 0$ we could make $U(f, P) - L(f, P) = \sum (M_k - m_k)\Delta x_k < \epsilon$.

    We did this by using the Extreme Value Theorem to show that M_k and m_k were values of f. We then used uniform continuity to show that we could make their difference as small as we wanted to by choosing a partition with small Δx_k's.

15. The real numbers are a complete ordered field. We used the fact that the real numbers formed a field in Abstract Algebra, and we may or may not have used the fact that they were ordered. We definitely did not use the fact that they were complete: every non-empty set of real numbers that is bounded above has a least upper bound.