

Scaling Graphs of Curves

to Demonstrate Behavior

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Introduction

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- The suggestion was to choose the ratio of horizontal and vertical axes in such a way that the average magnitude of the slope of the segments connecting adjacent points was 1. The process of choosing a ratio to match a given average magnitude is known as *banking*, and the text suggested banking to an angle of 45° , i.e., an average magnitude of slope of 1.

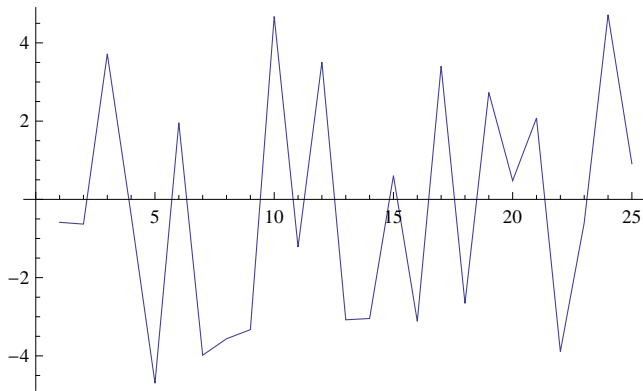
Introduction

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- The suggestion was to choose the ratio of horizontal and vertical axes in such a way that the average magnitude of the slope of the segments connecting adjacent points was 1. The process of choosing a ratio to match a given average magnitude is known as *banking*, and the text suggested banking to an angle of 45° , i.e., an average magnitude of slope of 1.
- I immediately started wondering what that would look like in the continuous case of plotting graphs of curves. This paper will explore how to do so and the implications for common curves.

Example: Data Points

Example

Data Points: Here is an example of some random data from a time series.



Scaling

- With this scaling, the peaks and valleys of the graph are all very sharp, making it harder to distinguish between them, or to tell when the values plateau. We can smooth it out by changing the scaling so that the average magnitude of the slope is 1.

Scaling

- With this scaling, the peaks and valleys of the graph are all very sharp, making it harder to distinguish between them, or to tell when the values plateau. We can smooth it out by changing the scaling so that the average magnitude of the slope is 1.
- We will need to model how the graph appears. Let (x_i, y_i) be the i th data point. Assign coordinates to the graph so that the lower left corner corresponds to the origin, and the upper right corner has coordinates (W, H) , where W and H are the width and height of the graph respectively. The *aspect ratio* is the ratio of the height to the width, i.e., H/W . Let x_{\min} , x_{\max} , y_{\min} and y_{\max} be the extremes of the data set. We will assume that the y_i are not all equal, and therefore that $y_{\min} < y_{\max}$.

Linear Transformation

We will linearly transform the coordinates of the data point to coordinates (x'_i, y'_i) in the graph:

$$x'_i = \left(\frac{x_i - x_{\min}}{x_{\max} - x_{\min}} \right) W$$
$$y'_i = \left(\frac{y_i - y_{\min}}{y_{\max} - y_{\min}} \right) H$$

Now x'_i measures what fraction of the way from x_{\min} to x_{\max} that x_i is, scaled by the width of the graph; similarly y'_i measures what fraction of the way from y_{\min} to y_{\max} that y_i is, scaled by the height of the graph.

Slopes

Let $\Delta x_i = x_{i+1} - x_i$ and similarly for the other variables.

$$\Delta x'_i = \frac{W \Delta x_i}{x_{\max} - x_{\min}}$$

$$\Delta y'_i = \frac{H \Delta y_i}{y_{\max} - y_{\min}}$$

$$\left| \frac{\Delta y'_i}{\Delta x'_i} \right| = \frac{H(x_{\max} - x_{\min})}{W(y_{\max} - y_{\min})} \left| \frac{\Delta y_i}{\Delta x_i} \right|$$

Computing the Aspect Ratio

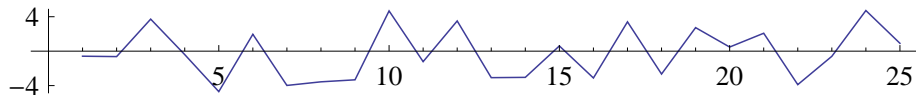
Now we apply our scaling criterion.

$$\begin{aligned} 1 &= \text{mean} \left| \frac{\Delta y'_i}{\Delta x'_i} \right| \\ &= \frac{H(x_{\max} - x_{\min})}{W(y_{\max} - y_{\min})} \text{mean} \left| \frac{\Delta y_i}{\Delta x_i} \right| \\ \frac{H}{W} &= \frac{y_{\max} - y_{\min}}{(x_{\max} - x_{\min}) \text{mean} \left| \frac{\Delta y_i}{\Delta x_i} \right|} \end{aligned}$$

Same Data, Scaled

Example

Same Data, Scaled: Here is an example of the same data, scaled so that the average magnitude of slope was 1.



While there are still many peaks and valleys in the graph, it is easier to pick out the most extreme ones and see where the data is more stable.

Graphs of Functions

- We will adjust our model for the graph of a function. Let (x, y) be a point on the graph of $y = f(x)$ where $a \leq x \leq b$ and $f(x)$ is differentiable. We will assume that $a < b$. Let m and M be the absolute minimum and maximum of $f(x)$ on $[a, b]$, which exist since $f(x)$ is continuous. (We will assume that $f(x)$ is non-constant, and therefore that $m < M$.)

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- Assign coordinates to the graph so that the lower left corner corresponds to the origin, and the upper right corner has coordinates (W, H) , where W and H are the width and height of the graph respectively. The aspect ratio is still the ratio of the height to the width, i.e., H/W .

Linear Transformation Again

We will linearly transform the coordinates of the points on the graph to coordinates (x', y') in the graph:

$$x' = \left(\frac{x - a}{b - a} \right) W$$

$$y' = \left(\frac{y - m}{M - m} \right) H$$

$$\begin{aligned} \left| \frac{dy'}{dx'} \right| &= \left| \frac{\frac{dy'}{dy} \frac{dy}{dx}}{\frac{dx'}{dx}} \right| \\ &= \frac{H(b - a)}{W(M - m)} \left| \frac{dy}{dx} \right| \end{aligned}$$

Computing the Aspect Ratio

We find the average by integrating and dividing by the width of the interval.

$$\begin{aligned} 1 &= \frac{H(b-a)}{W(M-m)} \left(\frac{1}{b-a} \right) \int_a^b \left| \frac{dy}{dx} \right| dx \\ &= \frac{H}{W(M-m)} \int_a^b \left| \frac{dy}{dx} \right| dx \\ \frac{H}{W} &= \frac{M-m}{\int_a^b \left| \frac{dy}{dx} \right| dx} \end{aligned}$$

Special Case: Increasing

There are two special cases where this simplifies tremendously.

- 1 $f(x)$ is increasing. Then $dy/dx \geq 0$ and we do not need the absolute values. By the Fundamental Theorem of Calculus:

$$\begin{aligned}\int_a^b \frac{dy}{dx} dx &= f(b) - f(a) \\ &= M - m \\ \frac{H}{W} &= 1\end{aligned}$$

Special Case: Increasing

- ② $f(x)$ is decreasing. Then $dy/dx \leq 0$ and the absolute values negate the derivative. By the Fundamental Theorem of Calculus:

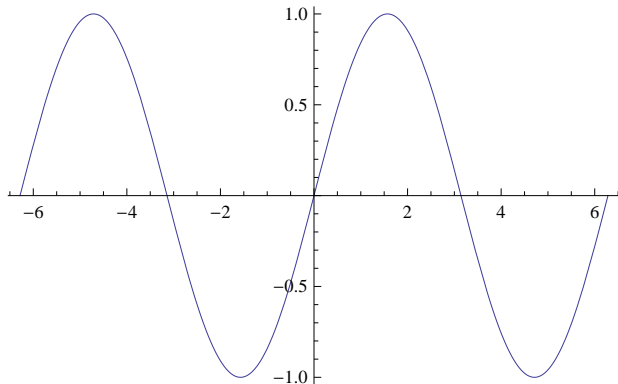
$$\begin{aligned}\int_a^b -\frac{dy}{dx} dx &= -(f(b) - f(a)) \\ &= -(m - M) \\ &= M - m \\ \frac{H}{W} &= 1\end{aligned}$$

To summarize: if $f(x)$ is monotone, draw the graph in a square.

Example: Unscaled Graph #1

Example

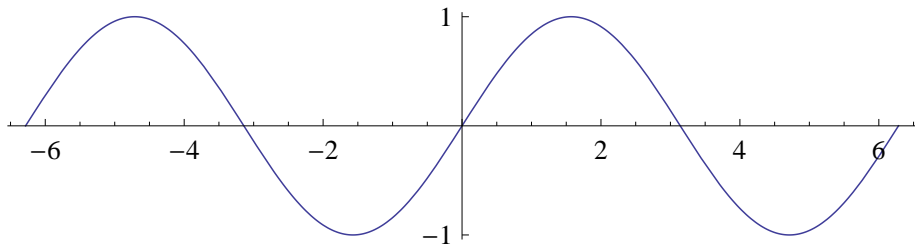
Graph of $y = \sin(x)$: Here is the default graph produced by *Mathematica* of $y = \sin(x)$ for $-2\pi \leq x \leq 2\pi$.



Example: Scaled Graph #1

Example

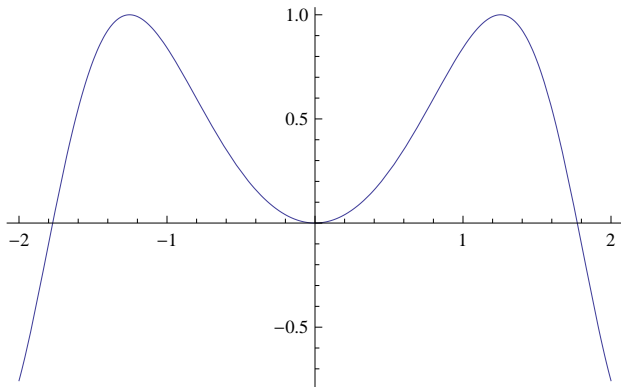
Graph of $y = \sin(x)$, scaled: Here is the same graph with the aspect ratio set by our criterion.



Example: Unscaled Graph #2

Example

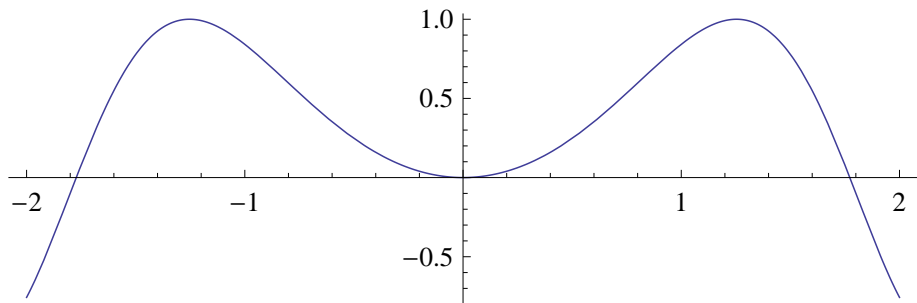
Graph of $y = \sin(x^2)$: Here's a graph of $y = \sin(x^2)$ for $-2 \leq x \leq 2$ with the default aspect ratio.



Example: Scaled Graph #2

Example

Graph of $y = \sin(x^2)$, scaled: Here is the same graph with the aspect ratio set by our criterion.



References



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Graphics Press LLC, Cheshire, CT, 2006.