Suppose retirees are paid \$1 for every year of their lives after they retire at age δ.

Let π be the amount paid in by each worker from ages γ through δ. This money should exactly cover the amount paid out to retirees.

Let b be the birth rate, i.e., the ratio of births to the mid-year population in a given year.

Let d be the death rate, i.e., the ratio of deaths to the mid-year population in a given year.

Let r be the growth rate, i.e., the difference between the birth b and death d rates.

A closed population is one without significant immigration or emigration.

Fertility profile refers to the distribution of female births to women in specific age categories. Let f_x be the ratio of female births to the number of women of age x in a given year.

Mortality profile refers to the distribution of deaths in specific age categories.

The Sharp-Lotka Theorem states that a closed population with a fixed fertility profile and a mortality profile will, regardless of its initial age distribution, eventually develop an asymptotically stable age distribution and eventually increase (or decrease) at a constant rate.

In a *stable population*, there is no significant immigration or emigration, and the fertility and the mortality profiles do not change over time. By the Sharp-Lotka theorem, a stable population will have as a limiting case constant values for b, d, and r.

Let (α, β) be the range of ages for women to bear children.

Let (γ, δ) be the range of ages for workers contributing to social security.

Let $S(x)$ be the fraction of females surviving to age x.

Let $F_x^t\, dx$ be the number of females at time t who are at ages between x and $x + dx$.

Let $B(t)\,dt$ be the number of female births between times t and $t + dt$.

$F_x^t\,dx = S(x)B(t-x)\,dx$ is the number of female births between times $t - x$ and $t - x + dx$ that survived for x years to time t.

$B(t) = \int_\alpha^\beta f_x F_x^t\,dx = \int_\alpha^\beta f_x S(x)B(t - x)\,dx$

With a fixed intrinsic rate of growth, $B(t-x) = e^{-rx}B(t)$.

$B(t) = \int_\alpha^\beta e^{-rx}B(t)S(x)f_x\,dx$

$\int_\alpha^\beta e^{-rx}S(x)f_x\,dx = 1$ can be used to solve for r given data on $S(x)$ and f_x.

The money paid in is

$$\pi \int_{\gamma}^{\delta} B(t-x) S(x)\, dx = \pi \int_{\gamma}^{\delta} B(t) e^{-rx} S(x)\, dx$$
$$= \pi B(t) \int_{\gamma}^{\delta} e^{-rx} S(x)\, dx,$$

where t is the current time. The first integral covers all the births x years ago for $\gamma \leq x \leq \delta$, taking into account the survival rate.

The money paid out is

$$\int_{\delta}^{\infty} B(t-x) S(x)\, dx = \int_{\delta}^{\infty} B(t) e^{-rx} S(x)\, dx$$
$$= B(t) \int_{\delta}^{\infty} e^{-rx} S(x)\, dx,$$

where t is the current time.

$$\pi = \frac{\int_{\delta}^{\infty} e^{-rx} S(x)\, dx}{\int_{\gamma}^{\delta} e^{-rx} S(x)\, dx}$$