EXPLORING FINITE TOPOLOGIES

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Abstract. A topology $\mathcal{T}$ on a set X is a collection of sets satisfying the following four requirements: a) $\emptyset \in \mathcal{T}$, b) the union of any collection of sets in $\mathcal{T}$ is contained in $\mathcal{T}$, c) the intersection of a finite collection of sets in $\mathcal{T}$ is contained in $\mathcal{T}$, and d) $X \in \mathcal{T}$. Using these axioms, it is possible to enumerate and categorize the possible topologies on a small finite set.

Some questions to explore:

- How fast does the number of topologies defined on n points grow?
- We can extend any permutation from S_n to the set of topologies. How many equivalence classes are there for a set of n points?
- What proportion of the topologies are Hausdorff? Regular? Normal?
- How often is a singleton its own closure?
- What is the connection between the closure of a point and the intersection of all of its neighborhoods?

1. Introduction

A topology is a collection of open sets for a given space [**?**]. There are four axioms for a topology on a space: it must contain the empty set, all unions of open sets, all finite intersections of open sets, and the space itself. Topologies are defined for spaces of interest, where the properties of closeness and continuity matter. How does the concept of a topology apply to a more trivial space, namely a finite space?

In what follows, a *finite topology* will refer to a topology on a finite space.

2. Construction

This construction of the collection of all topologies on a finite set is recursive.

For a given topology $\mathcal{T}$ and a set $S \subseteq X$, define $< \mathcal{T}, S >$ to be the coarsest topology finer than $\mathcal{T}$ that contains S as an open subset. This new topology is found by initially adjoining S to $\mathcal{T}$, and then recursively adding all intersections and unions of pairs of sets in $\mathcal{T}$. This process will terminate in a bounded number of steps, since $\mathcal{T} \subseteq \mathcal{P}(X)$, which is finite.

Let $C(X)$ be the coarse topology, containing only $\emptyset$ and X. Let T be a set of topologies (a "to-do" stack) to be refined, D be a set of topologies already done. Set $T = \{C(X)\}$ and $D = \emptyset$ initially.

While T is nonempty, remove an element $\mathcal{T}$. If it is already in D, continue. If not, add it to D, then find all topologies $< \mathcal{T}, S >$ as S ranges over the elements of $\mathcal{P}(X)$ and add them to T.

A topology, being a set of sets, is a subset itself of the power set $\mathcal{P}(X)$ of the space X. Each topology is forced to contain the empty set and the entire space. The set of all topologies is a subset itself of $\mathcal{P}(\{\mathcal{T} \in \mathcal{P}(X) \mid \emptyset, X \in \mathcal{T}\})$ and thus is

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bounded in size by $2^{(2^n-2)}$, where n is the number of points in X. The algorithm is thus guaranteed to terminate in a large but bounded number of iterations.

I have been able to examine spaces of size at most 5 using this algorithm. Code in `C++` for doing so is available at
`http://frodo.elon.edu/perl/parser.cgi/presentations.dat`.

3. Size of Computations

The number of topologies on a set of size n is bounded above by $2^{(2^n-2)}$. This number grows huge quickly. How accurate is this bound?

n	Number of Classes	$2^{(2^n-2)}$
1	1	1
2	4	4
3	29	64
4	355	16,384
5	6942	1,073,741,824

Clearly this upper bound is only meaningful for the first several cases. The number of topologies seems to grow exponentially (as opposed to the "doubly" exponential growth of the upper bound), but it is difficult to discern a pattern with only five data points.

4. Equivalence Classes of Topologies

A permutation σ in the symmetric group S_n induces a permutation of the topologies on X, simply by applying σ to every point in every open set. The orbits of this operation are equivalence classes of topologies, where two topologies are equivalent if one can be obtained by re-labeling the points of the other.

How many equivalence classes are there for a given n?

n	Number of Equivalence Classes	Number of Classes	Average Orbit Size
1	1	1	1.000
2	3	4	1.333
3	9	29	4.333
4	33	355	10.758
5	139	6942	49.942

The number of equivalence classes appears to follow an exponential curve.

5. Separation Properties

Note that in a finite topology, every point has a minimal open neighborhood, namely the (finite) intersection of every open set containing the point.

If a finite topology is T_1, then the minimal open neighborhood U for a given point p must contain only the point itself. If the neighborhood contained any other point q, then there must be an open set containing p and not q, contradicting the fact that U is a minimal open neighborhood of p.

Therefore the only finite Hausdorff topologies are the discrete ones, and similarly the only regular and normal topologies are discrete. The proportion of finite topologies that are Hausdorff/regular/normal is the reciprocal of their number.

6. Dual Topologies

The complements of open sets are closed sets. The axioms for closed sets are dual to those of open sets: the empty set must be closed, all intersections of closed sets must be closed, all finite unions of closed sets must be closed, and the space itself must be closed.

For a finite space, these axioms are identical, and thus for any given topology $\mathcal{T}$, the set of closed sets forms another topology, $\mathcal{T}^*$.

7. Minimal Neighborhoods of Points

Since there are duals for each topology, every open set in one topology corresponds to a closed set in the dual topology and vice versa.

Therefore the minimal open neighborhood in one topology corresponds to the closure of the point in the dual topology.

How large are the one point closures? The following table gives the relative frequencies of the sizes of a one point closure, averaged over all of the topologies for X.

	Relative Frequencies (%)				
n	1	2	3	4	5
1	100				
2	50	50			
3	41.379	34.483	24.138		
4	36.620	30.423	20.282	12.676	
5	33.679	27.370	19.188	12.561	7.203

8. Conclusion

There are many open questions related to finite topologies. Computing technology now permits us to find results for small space sizes. I hope that we will be able to find combinatoric expressions for many of these results in the near future.

References

[1] John G. Hocking and Gail S. Young, *Topology*, Dover Publications, Inc., 1961.