FILTERS FOR BEGINNERS

ABSTRACT. Filters arise in the discussion of such varied areas of mathematics as topology, the theory of uniform spaces, and the construction of the hyperreal numbers. They are a unifying concept, and their elementary applications are accessible to undergraduates. I will discuss examples of their use as a topic in the mathematics curriculum.

1. INTRODUCTION

Filters are a basic concept from set theory, used to generalize from the notion of a single point to that of a cluster of points that can not be distinguished. Since set theory is the foundation for so much of mathematics, filters prove to be a useful concept in many different areas.

Definition 1. A *filter* $\mathcal{F}$ on a set X is a collection of nonempty subsets of X such that:

1. The intersection of any two subsets in $\mathcal{F}$ is again in $\mathcal{F}$.
2. If A is in $\mathcal{F}$ and $A \subseteq B \subseteq X$ then B is also in $\mathcal{F}$.

The most important examples in what follows are:

1. Let x be a point in X. Define $\mathcal{F}_x$ to be the set of all subsets that contain x. Then $\mathcal{F}_x$ is a filter.
2. A subset of $\mathbb{N}$ is *cofinite* if its complement is finite. Let $\mathcal{F}_{\text{cof}}$ be the collection of subsets of $\mathbb{N}$ that are *cofinite*. Then $\mathcal{F}_{\text{cof}}$ is a filter.

In the first example, there is a natural connection between the filter $\mathcal{F}_x$ and the point x. In general, a filter will be connected to a collection of points that are associated. On the other hand, the second example points to some sort of limiting condition—a set S is in the filter if it contains all natural numbers from a certain point onward. (If k is the largest number in the complement of S, then S contains every number larger than k.) This limiting condition can be used in many different contexts related to the convergence of sequences.

2. Topology

Topology is often defined with respect to a given *metric* or distance function:

Definition 2. A *metric* is a function $d\colon X \times X \to X$ satisfying the following three conditions for all x, y, and z in X:

1. $d(x, y) \geq 0$ with equality only when $x = y$.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, x)$, known as the *triangle inequality*.

The ordered pair (X, d) is called a *metric space*.

The metric d is used to define a ball (whether or not X is a Euclidean space): a *ball* of radius $r \in \mathbb{R}$ with center $x \in X$ is the set of points whose distance from x is less than r. (This is also known as an open ball.)

Given a metric d, we can divide the points of any subset S of X into three types:

1. Interior points of S, each of which is the center of a ball contained entirely in S. Interior points of S are always contained in S.
2. Boundary points of S, such that every ball centered at such a point contains points both in S and in its complement. Boundary points of S are sometimes but not always contained in S.

3. Exterior points of S, each of which is the center of a ball contained entirely in the complement of S. Exterior points of S are never contained in S.

We can now define an open subset S of X to be one whose points are all interior points. Another way of phrasing the same concept is to say that S does not contain any of its boundary points.

Definition 3. A *topology* $\mathcal{T}$ on a set X is the collection of open subsets of X. These open subsets can be defined (without referring to a metric on X) as any collection $\mathcal{T}$ of subsets of X satisfying the following three conditions:

1. $\mathcal{T}$ contains both X and the empty set.
2. The intersection of any two subsets in $\mathcal{T}$ is again in $\mathcal{T}$.
3. The union of any collection of subsets in $\mathcal{T}$ is again in $\mathcal{T}$.

The ordered pair $(X, \mathcal{T})$ is called a *topological space.*

Another way of describing the topology of a set X uses *neighborhoods* of points instead of open subsets.

Definition 4. A neighborhood N_x of x is a subset of X containing an open set U_x that contains x.

Thus a neighborhood of x contains not only x but also points that are "near" x. The smaller the neighborhood, the nearer the points are to x. The intersection of any two neighborhoods of x is still a neighborhood of x, and a superset of a neighborhood of x is also a neighborhood of x. Thus the collection $\mathcal{N}_x$ of neighborhoods of a given point x form a filter.

Definition 5. A *neighborhood system* of X is the set $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ of all neighborhood filters. In an abuse of terminology, we shall refer to $(X, \mathcal{N})$ as a topological space.

There is a one-to-one correspondence between topologies and neighborhood systems. Given a topology $\mathcal{T}$ on a space X we can use it as above to define a neighborhood system $\mathcal{N}$. Conversely, given a neighborhood system $\mathcal{N}$, we can use it to recover the topology on X in the following manner: a set U is open if and only if it is a neighborhood of all of its points.

Since a topology can be defined axiomatically, it is only reasonable to expect that there is an abstract definition of a neighborhood system:

Definition 6. $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ is a neighborhood system for a set X if

1. $\mathcal{N}_x$ is a filter for every x in X.
2. Every S in $\mathcal{N}_x$ contains x.
3. For every U in $\mathcal{N}_x$, there is a V in $\mathcal{N}_x$ such that U is a neighborhood of every point in V, i.e., U is in $\mathcal{N}_y$ for every y in V. (V serves as the interior of U.)

If we take this definition as a starting point, we can define continuity and convergence in terms of either a metric or a neighborhood system.

Definition 7. A function $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *continuous* at $a \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \epsilon$. (This assumes that $f(a)$ and $f(x)$ are defined.) The idea is that we can force $f(x)$ to be arbitrarily close to $f(a)$ by choosing x to be close to a. If f is continuous at every a in X, then f is continuous.

In terms of a neighborhood filter:

Definition (7′)**.** A function $f\colon X \to Y$ between topological spaces $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$ is *continuous* at $a \in X$ if for every neighborhood V in $\mathcal{N}_{f(a)}$ there is a neighborhood U in $\mathcal{N}_a$ such that $f(U) \subseteq V$.

One of the fundamental concepts in topology is that of a *convergent sequence.*

Definition 8. In a metric space, we say that a sequence $\{x_n\}$ *converges* to a *limit* x_0 if for any $\epsilon > 0$, there is an N such that for n larger than N we have $d(x_n, x_0) < \epsilon$. In other words, by going out far enough in the sequence, we can get as close to the limit as we want.

Without a metric, we can phrase convergence in terms of filters. The sequence $\{x_n\}$ is a function x from $\mathbb{N}$ to X.

Definition (8′)**.** We say that x_n converges to the limit x_0 if for every U in $\mathcal{F}_{x_0}$ there is an S in $\mathcal{F}_{\text{cof}}$ such that $x(S) \subseteq U$. Again, by going out far enough in the sequence, i.e., by choosing an S with a large enough complement, we can get the elements of $x(S)$ into an arbitrarily small neighborhood of x_0.

[1] and [2] contain a thorough discussion of topology in terms of filters.

3. Uniformities

Uniform continuity, Cauchy sequences, and completeness are usually defined using a metric $d(x, y)$ on a given space X, since these concepts refer to properties of arbitrary pairs of points from X; a topology $\mathcal{T}$ on X does not contain enough information on its own. In order to define these three concepts without reference to a metric function, we need to refer to a collection of subsets of $X \times X$ instead of just X.

Notation.

1. Δ is the diagonal set of X, i.e., $\Delta = \{(x, x) \mid x \in X\}$.
2. If A is a subset of $X \times X$, then $A^{-1} = \{(y, x) \mid (x, y) \in A\}$.
3. If A and B are subsets of $X \times X$, then AB is defined to be equal to $\{(x, z) \mid \exists y \in X$ such that $(x, y) \in A$ and $(y, z) \in B)\}$. In particular, $A^2 = AA$.

Definition 9. A *uniformity* is a filter $\mathcal{F}$ defined on the set $X \times X$ satisfying the following properties:

1. Every subset U in $\mathcal{F}$ must contain the diagonal set Δ. (This is analogous to requiring $\mathcal{N}_x$ to contain x.)
2. If U is in $\mathcal{F}$ then so is U^{-1}. (Collectively, if not individually, the entourages are symmetric.)
3. For every U in $\mathcal{F}$ there is a V in $\mathcal{F}$ such that $V^2 \subseteq U$. (This requirement is analogous to the triangle inequality for metrics.)

The elements U of $\mathcal{F}$ are referred to as *entourages.* A *uniform space* is an ordered pair $(X, \mathcal{F}_X)$ where $\mathcal{F}_X$ is a uniformity defined on the space X.

Instead of using a metric to describe how close two points x and y are in terms of a real number, we can use the entourages to describe how close they are: x and y are U-close if (x, y) is in U, where U is an entourage.

If a metric d is given for a space X, it determines a uniformity $\mathcal{F} = \{\mathcal{F}_\epsilon \mid \epsilon > 0\}$ where $\mathcal{F}_\epsilon = \{(a, b) \in X \times X \mid d(a, b) < \epsilon\}$.

A uniformity $\mathcal{F}$ determines a neighborhood system $\mathcal{N} = \{\mathcal{N}_x \mid x \in X\}$ where $\mathcal{N}_x = \{U(x) \mid U \in \mathcal{F}\}$ with $U(x) = \{y \in X \mid (x, y) \in U\}$.

Let d_X and d_Y be metrics defined on X and Y respectively:

Definition 10. A function $f\colon X \to Y$ is *uniformly continuous* if for any $\epsilon > 0$, there is a δ such that $d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \epsilon$. In other words, we can force the outputs from f to be arbitrarily close if we force the inputs to f to be close to each other. This is fundamentally a concept involving arbitrary pairs of points, as opposed to an arbitrary point as in the definition of a continuous function.

The same definition, without metrics, reads as follows:

Definition (10′)**.** A function f from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$ is *uniformly continuous* if for any entourage V in $\mathcal{F}_Y$ there is an entourage U in $\mathcal{F}_X$ such that $f(U) \subseteq V$. Here $f(U) = \{(f(a), f(b)) \mid (a, b) \in U\}$.

With a metric d, we have:

Definition 11. A sequence $\{x_n\}$ is a *Cauchy sequence* if for any $\epsilon > 0$ there is an N such that whenever m and n are both larger than N, $d(x_m, x_n) < \epsilon$. In other words, we can force the elements of the sequence to be arbitrarily close to each other if we go out far enough.

The same definition, without a metric, becomes the following:

Definition (11′)**.** A sequence $\{x_n\}$ in the uniform space $(X, \mathcal{F}_X)$ is a *Cauchy sequence* if for any entourage U in $\mathcal{F}_X$ we can find an element S of $\mathcal{F}_{\text{cof}}$ such that (x_m, x_n) is in U for any m and n in S. The filter element S serves the role of N from the metric definition, and the entourage U is analogous to the requirement of the two points being within ϵ of each other.

Definition 12. A *complete* metric or uniform space is one in which all Cauchy sequences have a limit.

[1] and [3] are good, thorough introductions to uniform spaces.

4. Hyperreals

The hyperreal numbers are an extension of the real number system that includes infinite numbers (numbers that are larger in magnitude than any real numbers) as well as infinitesimals (nonzero numbers that are smaller in magnitude than any positive real number). They are constructed as equivalence classes of sequences of real numbers; the equivalence class of $\{1, 2, 3, \dots\}$ will be an infinite number, while the equivalence class of $\{1, 1/2, 1/3, \dots\}$ will be an infinitesimal number. Operations on these numbers will be defined by their actions on each component.

The trick is to be careful about how we define two sequences to be equivalent. We do not want to require convergence of any kind, or we'll end up with another version of the real numbers.

On the other hand, we want the properties of the sequences to depend on their end behavior and not how they start. Therefore, if two sequences differ by a finite number of elements, we will want them to be equivalent. This leads to one possible definition: $\{x_n\} \equiv \{y_n\}$ if and only if $\{n \mid x_n = y_n\} \in \mathcal{F}_{cof}$.

Unfortunately, such a definition would leave numbers like the equivalence class of $\{0, 1, 0, 1, \dots\}$ ambiguous. Is it 0? Is it 1? It satisfies the equation $x^2 - x = 0$.

First, we need a new definition. For an arbitrary set X, the filters on X can be ordered by inclusion. By an application of Zorn's lemma, any filter $\mathcal{F}$ is contained in a maximal filter $\mathcal{M}$. Since $\mathcal{M}$ contains as many subsets of X as possible, for every subset $S \subseteq X$, either S or its complement is contained in $\mathcal{M}$.

Definition 13. A maximal filter is called an *ultrafilter.*

Every filter is contained in an ultrafilter, but in general the ultrafilter will not be unique.

Notation. Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{F}_{\text{cof}}$. Let $[x_n]$ be the equivalence class of the sequence $\{x_n\}$.

Definition 14. The hyperreal numbers $\mathbb{R}^*$ are the equivalence classes of real sequences, where $\{x_n\} \equiv \{y_n\}$ if and only if $\{n \mid x_n = y_n\}$ is contained in $\mathcal{U}$. Any operation or relation defined on the real numbers can extended to the hyperreals by applying it to the components of the sequences.

For example:

1. $[x_n] + [y_n] = [x_n + y_n]$.
2. $[x_n] < [y_n]$ is true if and only if $\{n \mid x_n < y_n\}$ is in $\mathcal{U}$.
3. We can identify $r \in \mathbb{R}$ with $[r_n]$ where $r_n = r$ for all n.
4. If $[x_n] \neq [0]$, then we can let $[y_n] = [x_n]^{-1}$ where

$$y_n = \begin{cases} 0 & x_n = 0 \\ x_n^{-1} & x_n \neq 0 \end{cases}$$

   Then $\{n \mid x_n y_n = 1\}$ is in $\mathcal{U}$, and $[x_n][y_n] = [1]$.
5. $[1, 2, 3, \ldots]$ is larger than any real number $[r]$, since $\{n \mid n > r\}$ is cofinite and thus in $\mathcal{U}$. Thus $[1, 2, 3, \ldots]$ is an infinite number.
6. $[1, 1/2, 1/3, \ldots]$ is greater than $[0]$ but less than $[r]$ for any positive real number r for a similar reason. It is an infinitesimal number.

REFERENCES

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3. Warren Page, *Topological uniform structures*, Dover, New York, 1988.