Using Infinite Numbers in Calculus

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March 27, 2002

Abstract

Nonstandard analysis is a technique for studying calculus and analysis using infinite and infinitesimal numbers rigorously. I will cover how this approach could be used in differential and integral calculus.

1 Introduction

Modern calculus and analysis is built around the notion of a limit: as x approaches a, what happens to $f(x)$? Since the nineteenth century we have formulated limits in terms of distances, i.e., δ's and ϵ's. It hasn't always been this way: Newton and Leibniz discussed derivatives as ratios of infinitesimal (infinitely small) numbers, and Euler used infinity on a regular basis.

In 1966, Abraham Robinson reformulated limits (and thus calculus and analysis) in terms of algebra by extending the real number system to include infinite and infinitesimal numbers in a rigorous fashion. The end result is a different way of studying calculus, which yields the same results but can be truer to the original way in which it was discovered.

Much of this paper was inspired by [2].

2 Uses for Infinite and Infinitesimal Numbers

An infinite number should be a number whose magnitude is greater than that of any real number. If we add an infinite number to the reals and maintain our normal rules for arithmetic, then we have to add a lot of infinities. Let ω be a positive infinite number. Then $\omega + 1$, $\omega + 2$, ω^2, and $-\omega$ will all have to be distinct numbers that are infinite.

If we have infinite numbers, then for any function $f(x)$ we could evaluate the limit at infinity by plugging in an infinite number, e.g., $f(\omega)$ and seeing if the result depends upon which infinite number we choose.

If we view integrals as limits of Riemann sums, then we can define them as functions of the number n of subintervals for a regular partition. Given a formula $R(n)$ for the Riemann sum, what happens if we put different infinities in R in place of n?

Once we have infinite numbers like ω, their reciprocals will be less in magnitude than the magnitude of any nonzero real number. Such infinitely small numbers are called *infinitesimals.* (Note that we will allow 0 to be called an infinitesimal number; otherwise in what follows we will constantly have to deal with it as a special case.)

If ι is an infinitesimal, then how does $f(a+\iota)$ behave? Its values should be the value of $f(x)$ for x "near" to a, which is what we normally think of as a limit.

Similarly as we take derivatives, we usually work with the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If ι is a nonzero infinitesimal number, we can look at

$$\frac{f(x+\iota) - f(x)}{\iota}$$

How does it depend on the choice of ι?

3 Definition of Hyperreals

We are trying to extend the real numbers to include infinite numbers. We will call the new set of numbers the *hyperreal number system.* A good strategy is to imitate how the reals are constructed from the rational numbers.

One way to go from the rational numbers to the reals is to look at sequences of rational numbers that have the corresponding real number as their limit. Since we can't use the reals to define the reals, we instead look at Cauchy sequences of rational numbers.

Since there can be many Cauchy sequences of rational numbers with the same real limit, we say that two sequences are equivalent if their difference has a limit of 0.

In constructing the hyperreals, we start by forming sequences $\{x_n\}$ of real numbers. We want to think of the "limit" of the sequence being the hyperreal created.

We can perform arithmetic on sequences pretty easily:

- $\{x_n\} + \{y_n\} = \{x_n + y_n\}$
- $\{x_n\} - \{y_n\} = \{x_n - y_n\}$
- $\{x_n\} \times \{y_n\} = \{x_n y_n\}$
- $\{x_n\} \div \{y_n\} = \{x_n / y_n\}$ where $y_n \neq 0$ for all n.

Unfortunately, most sequences of real numbers don't have limits as we normally think of them. We aren't placing any restrictions on the sequences, so $\{1, 2, 3, \ldots\}$ is fair game. We also can't divide by many sequences (if any of their elements are 0); usually we only forbid division by a single number 0.

We need to construct an equivalence relation that will allow us to only have one 0. One way of doing so is by saying that two sequences $\{x_n\}$ and $\{y_n\}$ are equivalent if $x_n = y_n$ for *almost all* n, whatever that means.

4 Filters

A *filter* F on a set S is a collection of nonempty subsets that satisfy two requirements:

1. If A and B are in F, then so is $A \cap B$.
2. If $A \in F$ and $A \subseteq B$, then B is in F.

Intuitively, a filter is a generalization of the idea of the neighborhoods of a point from topology. Every point $x \in S$ has a point filter $F = \{U \subseteq S \mid x \in U\}$; in a topological space x has a neighborhood filter $F = \{U \subseteq S \mid \exists V \subseteq S \ni x \in V \subseteq U, V \text{ open}\}$.

An example that we will be using is a filter on the natural numbers called the *finite complement filter* on $\mathbb{N}$: it consists of all the complements of finite sets.

Filters can be ordered by inclusion: if F and G are two filters on S, then $F \leq G$ if and only if every set in F is in G.

A maximal filter is called an *ultrafilter.* An ultrafilter has as many sets in it as it can without having the empty set, which would come from the requirement that it be closed under intersections. It can be shown that for an ultrafilter F and an arbitrary subset $T \subseteq S$, either $T \in F$ or $S - T \in F$.

Using Zorn's Lemma, it is possible to show that every filter is contained within an ultrafilter.

We are finally ready to define *almost always*: let F be a fixed ultrafilter on $\mathbb{N}$ containing the finite complement filter. Then any property $P(n)$ defined on the natural numbers will be *almost always* true if $\{n \mid P(n) \text{ is true}\}$ is contained in F.

It follows from the fact that since F is an ultrafilter, for any property $P(n)$, either $P(n)$ or its negation $\neg P(n)$ is almost always true.

5 Equivalence Classes and Extensions

Define two sequences of real numbers $\{x_n\}$ and $\{y_n\}$ to be equivalent if $x_n = y_n$ almost everywhere. It is tedious but easy to see that this is an equivalence relation from the definition of the finite complement filter.

Define the *hyperreal numbers* $\mathbb{R}^*$ to be the equivalence classes determined by this equivalence relation. More generally if $S \subseteq \mathbb{R}$ then S^* is the equivalence classes of sequences of elements of S. Note that we have the hypernatural numbers $\mathbb{N}^*$, the hyperintegers $\mathbb{Z}^*$, and the hyperrational numbers $\mathbb{Q}^*$.

We will denote an equivalence class containing $\{x_n\}$ by $[x_n]$.

For any function $f : D \to R$ where D and R are subsets of $\mathbb{R}$, define $f^* : D^* \to R^*$ by $f^*[x_n] = [f(x_n)]$.

We'll skip over most proofs, but let's actually do this one: f^* is well-defined. Suppose that $[x_n] = [y_n] \in D^*$, i.e., that $x_n, y_n \in D$ for all n and that $x_n = y_n$ almost always. Then $\{n \mid x_n = y_n\} \subseteq \{n \mid f(x_n) = f(y_n)\}$, and thus since the superset of an element of a filter is also in the filter, $f(x_n) = f(y_n)$ almost always and $[f(x_n)] = [f(y_n)]$.

Let R be a relation defined on $\mathbb{R}$. Then define R^* on $\mathbb{R}^*$ by $[x_n]R^*[y_n]$ if and only if $x_n R y_n$ is true almost always. Note that R could be $<$ or $>$.

6 The Hyperreals Are an Ordered Field Extension of the Reals

Every real number r corresponds to the equivalence class of the constant sequence $\{r_n\}$, where $r_n = r$ for all n. We shall refer to this class as $[r]$.

We define the arithmetic operations:

- $[x_n] + [y_n] = [x_n + y_n]$
- $[x_n] - [y_n] = [x_n - y_n]$
- $[x_n][y_n] = [x_n y_n]$
- $[x_n]/[y_n] = [x_n/y'_n]$ where $[y_n] \neq [0]$.

In the definition of division, if $[y_n] \neq [0]$, then $y_n \neq 0$ almost always. Then define $y'_n = y_n$ if $y_n \neq 0$, and $y'_n = 1$ if $y_n = 0$. We have that $y'_n = y_n$ almost always, and $[y'_n] = [y_n]$.

The field axioms hold on $\mathbb{R}^*$ since they hold for each element of each sequence of real numbers, and thus their equivalence classes.

In order to have an ordered field, we need to be able to split $\mathbb{R}^*$ into negative, zero, and positive numbers. For any $[x_n] \in \mathbb{R}^*$, either $x_n \geq 0$ or $x_n < 0$ almost always. (This is one reason we needed to use an ultrafilter to define almost always). If $x_n < 0$ almost always, then $[x_n]$ is a negative number. If $x_n \geq 0$ almost always, then either $x_n > 0$ or $x_n = 0$ almost always. These two cases correspond to the positive numbers and zero.

It is easy to verify that the sum and the product of two positive hyperreal numbers is positive, and thus the hyperreals are an ordered field extension of the reals.

7 Infinite and Infinitesimal Numbers

We are now ready to define infinite and infinitesimal numbers, which are the motivation for creating the hyperreal numbers.

An *infinite* number is one whose magnitude is greater than any real number. An example is $\omega = [1, 2, 3, \ldots]$; for any real number $[r]$, $[1, 2, 3, \ldots] > [|r|]$ since it is true almost always in terms of coordinates. Any number that is not infinite is *finite*.

An *infinitesimal* number is one whose magnitude is less than that of any positive real number; note that by this definition 0 is an infinitesimal number. $\iota = [1, 1/2, 1/3, \ldots] = 1/\omega$ is a positive infinitesimal number.

8 Orbits and Standard Parts

As soon as we add infinitesimal numbers, we make all of our numbers fuzzy: now for every real number there are a bunch of numbers that are infinitely close to them.

Two hyperreal numbers are *infinitely close* if their difference is an infinitesimal. Since the sum of any two infinitesimal numbers is infinitesimal, and the negative of an infinitesimal is infinitesimal, the relation *infinitely close* is an equivalence relation.

Every finite hyperreal number is infinitely close to a unique real number.

Uniqueness is easy: if h is infinitely close to real numbers r and s, then r and s are infinitely close. But since their difference is real, and the only real infinitesimal number is 0, $r - s = 0$ and $r = s$.

Existence takes a little more work. Let h be a finite hyperreal number. Since it is finite, there must be at least one real number greater than h. Let S be the set of real numbers less than h. S is bounded above; let r be its least upper bound.

If $r = h$, we are done.

If $r > h$, then $r - h$ must be infinitesimal, or else there would be a real number s such that $r - h > s > 0$, $r - s > h$, and $r - s$ is a smaller upper bound for S, a contradiction.

If $r < h$, then $h - r$ must be infinitesimal, or else there would be a real number s such that $h - r > s > 0$, $r + s < h$, $r + s \in S$, and r would not be an upper bound for S, a contradiction.

Note that while we use the fact that $\mathbb{R}$ has least upper bounds for every bounded set, $\mathbb{R}^*$ does not. For example, the set of all infinitesimal numbers is bounded but has no least upper bound.

For every finite hyperreal number h, define $\text{st}(h)$, the standard part of h, to be the real number that is infinitely close to h.

We have the following properties for $\text{st}(h)$:

- If h_1 and h_2 are both finite, then $\text{st}(h_1 + h_2) = \text{st}(h_1) + \text{st}(h_2)$.
- If ι is infinitesimal, then $\text{st}(\iota) = 0$. If h is any finite number, then $\text{st}(h\iota) = 0$.

9 Limits

For any real function $f(x)$ and any real number a, we want the limit of $f(x)$ as $x \to a$ to be a number L if whenever x is close to a (but not equal to a), $f(x)$

is close to L.

Define $\lim_{x\to a} f(x) = L$ if and only if $\mathrm{st}(f^*(a+\iota)) = L$ for all nonzero infinitesimal numbers ι.

For example: if $f(x) = x^2$ and $a = 3$, $\mathrm{st}(f^*(3+\iota)) = \mathrm{st}(9+6\iota+\iota^2) = 9$, and $\lim_{x\to 3} x^2 = 9$.

On the other hand: if $f(x) = |x|/x$ and $a = 0$, then $\mathrm{st}(f^*(\iota)) = \pm 1$ depending on whether ι is positive or negative, and thus $\lim_{x\to 0} |x|/x$ has no limit.

Similarly, define $\lim_{x\to\infty} f(x) = L$ if and only if $\mathrm{st}(f^*(I)) = L$ for every positive infinite number I.

10 Continuity

As usual, a function $f(x)$ is continuous at a point a if its value and its limit there are the same: $f(x)$ is continuous at $x = a$ if and only if $\mathrm{st}(f^*(a+\iota)) = f(a)$ for all infinitesimal numbers ι.

11 Derivatives

We define $f'(x) = m$ if and only if

$$\mathrm{st}\left(\frac{f(x+\iota)-f(x)}{\iota}\right) = m$$

for all nonzero infinitesimals ι.

For example, if $f(x) = x^3$, then

$$\begin{aligned}
\mathrm{st}\left(\frac{x^3+3x^2\iota+3x\iota^2+\iota^3-x^3}{\iota}\right) &= \mathrm{st}\left(\frac{3x^2\iota+3x\iota^2+\iota^3}{\iota}\right) \\
&= \mathrm{st}\left(3x^2+3x\iota+\iota^2\right) \\
&= 3x^2
\end{aligned}$$

Another way of defining $f'(x)$ is that for any infinitesimal ι, there is an infinitesimal ι_2 such that $f(x+\iota) = f(x) + f'(x)\iota + \iota\iota_2$. This form of the definition works nicely in proving some of the main differentiation rules.

For example: assume that $f(x)$ and $g(x)$ are differentiable. Then $f(x+\iota) = f(x) + f'(x)\iota + \iota\iota_2$ and $g(x+\iota_3) = g(x) + g'(x)\iota_3 + \iota_3\iota_4$ where each ι is an infinitesimal number.

Therefore

$$\begin{aligned}
f(g(x+\iota)) &= f(g(x)+g'(x)\iota+\iota\iota_2) \\
&= f(g(x)) + f'(g(x))(g'(x)\iota+\iota\iota_2) + (g'(x)\iota+\iota\iota_2)\iota_3 \\
&= f(g(x)) + f'(g(x))g'(x)\iota + \iota(f'(g(x))g'(x)\iota 2 + g'(x)\iota_3 + \iota_2\iota_3) \\
&= f(g(x)) + f'(g(x))g'(x)\iota + \iota\iota_4
\end{aligned}$$

and $(f(g(x))' = f'(g(x))g'(x)$.

Similarly for the product rule:

$$\begin{aligned}
f(x+\iota)g(x+\iota) &= (f(x)+f'(x)\iota+\iota\iota_2)(g(x)+g'(x)\iota+\iota\iota_3) \\
&= f(x)g(x)+(f'(x)g(x)+f(x)g'(x))\iota+\iota(f(x)\iota_3 \\
&\quad +f'(x)g'(x)\iota+g(x)\iota_2) \\
&= f(x)g(x)+(f'(x)g(x)+f(x)g'(x))\iota+\iota\iota_4
\end{aligned}$$

and thus $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$.

12 Integration

For a continuous function $f(x)$ defined on the interval $[a,b]$, the definite integral $\int_a^b f(x)\,dx$ is the limit of a Riemann sum $R(n) = \sum_{i=1}^n f(t_i)\Delta x$ as the number of terms in the sum goes to infinity, no matter how the points $t_i \in [x_{i-1}, x_i]$ in the sum are chosen. (As usual, $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$.) For our purposes fix $t_i = (x_{i-1}+x_i)/2$.

In nonstandard analysis, we define $\int_a^b f(x)\,dx = \mathrm{st}(R^*(N))$ for infinite N. It is an interesting question as to what other functions are so integrable, and how the choice of t_i's affects convergence.

13 Conclusion

Nonstandard analysis is another perspective on calculus and analysis. It provides a rigorous framework for our intuitive notions of infinity and infinitesimal numbers.

14 References

This paper can be obtained electronically from the following address:
`http://frodo.elon.edu/presentations/nonstandardpaper.pdf`

References

[1] Joseph Warren Dauben, *Abraham Robinson*, Princeton University Press, 1995.

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[3] R. F. Hoskins, *Standard and Nonstandard Analysis*, Ellis Horwood, 1990.

[4] Albert E. Hurd, Peter A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, 1985.