

# A Brief Introduction to Term Rewriting

Jeffrey Clark  
Elon University

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# Don't Take Too Many Notes

This talk is available from my web-site:

`http://frodo.elon.edu`

under the link **Presentations**.

# Introduction: Expressions

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- The class of equivalent expressions can be ordered by complexity.
- If out of all the equivalent expressions for expressing the same concept there is a simplest one, that form is to be preferred.

## Introduction: Examples

- Positive fractions  $a/b$  and  $c/d$  are equivalent if  $ad = bc$ ; for equivalent fractions, the smaller the magnitude of the numerator and/or denominator the better. The simplest form of the fraction has the smallest magnitude for both the numerator and denominator.

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- Two polynomial expressions are equivalent if the total of the coefficients for each power of the variable are equal; for equivalent expressions the smaller the number of terms the simpler the expression, and if the terms are the same then ordering them from highest to lowest power is simpler; the standard form for a polynomial is the simplest.

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- Terms are written many different ways:  $x_1 + x_2$ ,  $D(f(x)/g(x))$ , with mixtures of prefix, infix, and postfix notation. The theory is easiest to express using prefix notation:  $+(x_1, x_2)$ ,  $D(/(f(x), g(x)))$ .

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- A relation  $R$  is an **equivalence relation** if it is reflexive, symmetric, and transitive.

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- Simplification involves trying to find the simplest member of an equivalence class if it exists.

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- A relation  $R$  is an **order relation** if it is irreflexive and transitive.
- Simplification (in various contexts) is an order relation.

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- Collect like terms:  $2x^2 + 3x^2 \rightarrow 5x^2$
- Distribute differentiation in sums:  
 $D(x^2 + x^3) \rightarrow D(x^2) + D(x^3)$

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- If defined simplification for fractions as moving towards larger denominators, then the simplification would never end.
- We require for simplification that there be no infinite chains of equivalent, simpler expressions.
- Such an ordering relation on equivalence classes is said to be **terminating**.
- Given any expression, its simplest equivalent form is called its **canonical form**.

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- Such an ordering relation on equivalence classes is said to be **confluent**.

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- The Sum Rule (when its hypotheses are satisfied) can be written as  $D(+ (x_1, x_2)) \rightarrow + (D(x_1), D(x_2))$ .
- A rewrite rule can be applied to a term  $t$  if any sub-term  $t_0$  of  $t$  is of the same form as the left side of the rewrite rule, with some substitution for the variables appearing in  $t_L$ . Then applying that same substitution to the variables in the right side of the rule yields a replacement for the sub-term  $t_0$ .

## Rewrite Rules: Example

- Applying the Sum Rule to  $D(+(+x_1, x_2), x_3)$  requires the substitution  $x_1 \rightarrow +(x_1, x_2)$  and  $x_2 \rightarrow x_3$ ; applying that substitution to  $+(D(x_1), D(x_2))$  yields  $+(D(+x_1, x_2), D(x_3))$ .

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- A second application of the Sum Rule yields  $+(+(D(x_1), D(x_2)), D(x_3))$ .

# Rewrite System

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- Rewrite systems, when terminating and confluent, are an efficient way to take an expression and simplify it to an equivalent, canonical form.

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- We can encode our rules for doing so in rewriting systems.

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