Final Exam Solutions

Math 321-A

Friday, May 16, 2003

1. We will be integrating over the region in the xy-plane bounded by the intersection of the two surfaces.

$$\begin{aligned} x^2 + 2xy + y^2 + 4 &= 2x^2 + 2xy + 2y^2 \\ 4 &= x^2 + y^2 \end{aligned}$$

The region is a disk of radius 2 centered at the origin. We will use cylindrical coordinates.

$$\begin{aligned} z &= x^2 + 2xy + y^2 + 4 \\ &= r^2 + 2r^2\cos(\theta)\sin(\theta) + 4 \\ z &= 2x^2 + 2xy + 2y^2 \\ &= 2r^2 + 2r^2\cos(\theta)\sin(\theta) \end{aligned}$$

We find the volume by integrating 1 over the solid region.

$$\begin{aligned} V &= \int\int\int_R 1\,dV \\ &= \int_0^{2\pi}\int_0^2\int_{2r^2+2r^2\cos(\theta)\sin(\theta)}^{r^2+2r^2\cos(\theta)\sin(\theta)+4} (1)r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 \left(4 - r^2\right) r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 \left(4r - r^3\right)\,dr\,d\theta \\ &= \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} 4\,d\theta \\ &= 8\pi \\ &= 25.133 \end{aligned}$$

2.

$$\begin{aligned}\vec{B} &= \frac{\vec{v}\times\vec{a}}{\|\vec{v}\times\vec{a}\|}\\ &= \frac{<3,-1,4>\times<2,5,-6>}{\|<3,-1,4>\times<2,5,-6>\|}\\ &= \frac{<-14,26,17>}{\|<-14,26,17>\|}\\ &= \frac{<-14,26,17>}{\sqrt{1161}}\\ &= \frac{<-14,26,17>}{3\sqrt{129}}\end{aligned}$$

3.

$$\begin{aligned}\int_C f\,ds &= \int_1^3 f(x(t),y(t))\|\vec{v}(t)\|\,dt\\ &= \int_1^3 f(t,t^2)\|<1,2t>\|\,dt\\ &= \int_1^3 t\sqrt{1+4t^2}\,dt\end{aligned}$$

Let $u = 1+4t^2$, $du = 8t\,dt$, $t\,dt = du/8$, and when $t = 1$, $u = 5$ while when $t = 3$, $u = 37$.

$$\begin{aligned}\int_C f\,ds &= \frac{1}{8}\int_5^{37} u^{1/2}\,du\\ &= \frac{1}{8}\frac{u^{3/2}}{3/2}\bigg|_5^{37}\\ &= \frac{37\sqrt{37}-5\sqrt{5}}{12}\\ &= 17.8235\end{aligned}$$

4.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\
&= \frac{\cos\theta e^{\sin\theta}\sin\theta + e^{\sin\theta}\cos\theta}{\cos\theta e^{\sin\theta}\cos\theta - e^{\sin\theta}\sin\theta} \\
&= \frac{\cos\theta\sin\theta + \cos\theta}{\cos^2\theta - \sin\theta} \\
\left.\frac{dy}{dx}\right|_{\theta=\pi/4} &= \frac{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2 - \frac{1}{\sqrt{2}}} \\
&= \frac{1+\sqrt{2}}{1-\sqrt{2}} \\
&= -3 - 2\sqrt{2} \\
&= -5.828
\end{aligned}
$$

5.

$$
\begin{aligned}
\vec{F}(x,y,z) &= < x\sin y\cos z, x\sin y\sin z, x\cos y > \\
&= < M, N, P > \\
\nabla \cdot \vec{F}(x,y,z) &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \\
&= \sin y\cos z + x\cos y\sin z + 0 \\
&= \sin y\cos z + x\cos y\sin z
\end{aligned}
$$

6. One reason is that there is a single equation for all conic sections in polar coordinates while there are three different equations for conic sections in Cartesian coordinates. Another reason is that the distance r from the origin at the sun played a key role in the inverse square law.

7.

$$
\begin{aligned}
r^2 + z^2 &= \frac{r}{z} \\
\rho^2\sin^2\phi + \rho^2\cos^2\phi &= \frac{\rho\sin\phi}{\rho\cos\phi} \\
\rho^2 &= \tan\phi
\end{aligned}
$$

8.

$$\begin{aligned}
\vec{r}(t) &= \left(t^2, t+5, t^3\right) \\
\vec{v} &= \left\langle 2t, 1, 3t^2 \right\rangle \\
\|\vec{v}\| &= \sqrt{9t^4 + 4t^2 + 1} \\
\vec{a} &= < 2, 0, 6t > \\
\vec{v} \times \vec{a} &= \left\langle 6t, -6t^2, -2 \right\rangle \\
\|\vec{v} \times \vec{a}\| &= \sqrt{36t^2 + 36t^4 + 4} \\
&= 2\sqrt{9t^4 + 9t^2 + 1} \\
\kappa &= \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3} \\
&= \frac{2\sqrt{9t^4 + 9t^2 + 1}}{\left(9t^4 + 4t^2 + 1\right)^{3/2}}
\end{aligned}$$

9. We will substitute the line's coordinates into the plane's equation, solve for t, and then find the point of intersection.

$$\begin{aligned}
x &= 2 - 3t \\
y &= 3 + 4t \\
z &= 4 - 5t \\
4x - y - 3z &= 12 \\
4(2 - 3t) - (3 + 4t) - 3(4 - 5t) &= 12 \\
-7 - t &= 12 \\
t &= -19 \\
< 2 - 3t, 3 + 4t, 4 - 5t >|_{t=-19} &= (59, -73, 99)
\end{aligned}$$

10. The center (h, k) of the ellipse is at the center of the four points, $(5, -2)$. a is the distance from the center to the east point, namely 3. b is the distance from the center to the north point, namely 2.

$$\begin{aligned}
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1 \\
\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} &= 1
\end{aligned}$$

11. Since we aren't given a parametrization for the curve between the two points, we will have to use the Fundamental Theorem of Line Integrals:

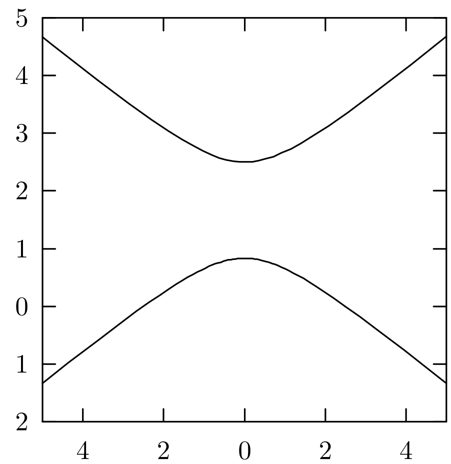
we will try to find a potential function, and evaluate it at the two points and take the difference.

$$
\begin{aligned}
\vec{F}(x,y) &=< y\cos(xy), x\cos(xy) > \\
&= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\
\frac{\partial f}{\partial x} &= y\cos(xy) \\
f(x,y) &= \sin(xy) + g(y) \\
\frac{\partial f}{\partial y} &= x\cos(xy) + g'(y) \\
&= x\cos(xy) \qquad \text{(from } \vec{F}\text{)} \\
g'(y) &= 0 \\
g(y) &= C \\
f(x,y) &= \sin(xy) + C \\
\int_C \vec{F} \cdot d\vec{r} &= f\left(\frac{\pi}{2}, 1\right) - f(1,\pi) \\
&= \sin\left(\frac{\pi}{2}\right) - \sin(\pi) \\
&= 1 - 0 \\
&= 1
\end{aligned}
$$

12. First we divide 2 out of the numerator and denominator to put the equation into a recognizable form.

$$
\begin{aligned}
r &= \frac{5}{2 + 4\sin\theta} \\
&= \frac{5/2}{1 + 2\sin\theta}
\end{aligned}
$$

The curve is a hyperbola with eccentricity 2.



13. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\begin{aligned}
\cos\theta &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} \\
&= \frac{<3,-2,6>\cdot<-1,1,3>}{\|<3,-2,6>\|\|<-1,1,3\|} \\
&= \frac{13}{\sqrt{49}\sqrt{11}} \\
&= \frac{13}{7\sqrt{11}} \\
\theta &= \cos^{-1}\left(\frac{13}{7\sqrt{11}}\right) \\
&= 0.976
\end{aligned}$$

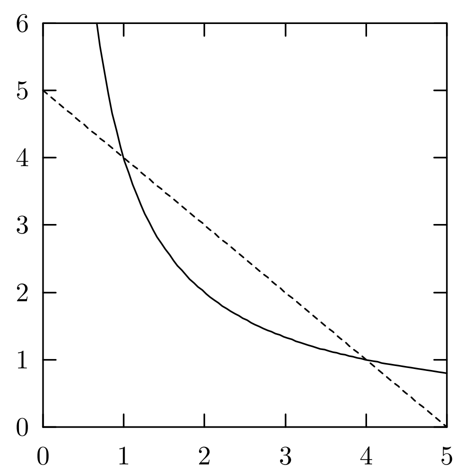
14.

$$\begin{aligned}
\text{proj}_{\vec{b}}\,\vec{a} &= \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b} \\
&= \frac{<2,-5,-1>\cdot<-3,2,1>}{<-3,2,1>\cdot<-3,2,1>} <3,2,1> \\
&= -\frac{17}{14} <-3,2,1>
\end{aligned}$$

15.

$$
\begin{aligned}
\vec{r}(t) &= \left(\cos\left(t^3\right), \sin\left(t^3\right)\right) \\
\vec{v}(t) &= \left(-3t^2 \sin\left(t^3\right), 3t^2 \cos\left(t^3\right)\right) \\
\|\vec{v}(t)\| &= \sqrt{\left(-3t^2 \sin\left(t^3\right)\right)^2 + \left(3t^2 \cos\left(t^3\right)\right)^2} \\
&= \sqrt{9t^4 \sin^2\left(t^3\right) + 9t^4 \cos^2\left(t^3\right)} \\
&= \sqrt{9t^4} \\
&= 3t^2 \\
\text{arc-length} &= \int_0^{\sqrt[3]{\pi}} \|\vec{v}(t)\|\, dt \\
&= \int_0^{\sqrt[3]{\pi}} 3t^2\, dt \\
&= t^3\Big|_0^{\sqrt[3]{\pi}} \\
&= \pi
\end{aligned}
$$

16. The graph of the curves has the line on top.



The curves intersect twice:

$$
\begin{aligned}
\frac{4}{x} &= 5 - x \\
4 &= 5x - x^2 \\
x^2 - 5x + 4 &= 0 \\
x &= 1, 4
\end{aligned}
$$

First we find the mass and the moment:

$$
\begin{aligned}
m &= \int_R \rho(x,y)\,dA \\
&= \int_1^4 \int_{4/x}^{5-x} k\,dy\,dx \\
&= k\int_1^4 y\big|_{y=4/x}^{y=5-x}\,dx \\
&= k\int_1^4 \left(5 - x - \frac{4}{x}\right)\,dx \\
&= k\left(5x - \frac{x^2}{2} - 4\ln x\right)\Bigg|_1^4 \\
&= k\left((20 - 8 - 4\ln 4) - \left(5 - \frac{1}{2} - 0\right)\right) \\
&= \left(\frac{15}{2} - 4\ln 4\right)k \\
&= 1.955k
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \int_R x\rho(x,y)\,dA \\
&= \int_1^4 \int_{4/x}^{5-x} kx\,dy\,dx \\
&= k\int_1^4 xy\big|_{y=4/x}^{y=5-x}\,dx \\
&= k\int_1^4 x\left(5 - x - \frac{4}{x}\right)\,dx \\
&= k\int_1^4 \left(5x - x^2 - 4\right)\,dx \\
&= k\left(\frac{5x^2}{2} - \frac{x^3}{3} - 4x\right)\Bigg|_1^4 \\
&= k\left(\left(40 - \frac{64}{3} - 16\right) - \left(\frac{5}{2} - \frac{1}{3} - 4\right)\right) \\
&= k\left(\frac{240 - 128 - 96 - 15 + 2 + 24}{6}\right) \\
&= \frac{27k}{6} \\
&= \frac{9k}{2}
\end{aligned}
$$

Now we find $\bar{x} = \bar{y}$:

$$\begin{aligned}
\bar{x} &= \bar{y} \\
&= \frac{M_y}{m} \\
&= \frac{9k/2}{(15/2 - 4\ln 4)k} \\
&= \frac{9}{15 - 8\ln 4} \\
&= \frac{9}{15 - 16\ln 2} \\
&= 2.302
\end{aligned}$$