Homework #9 Solutions

Math 312-A

Due Tuesday, April 17, 2007

page 272, #1–4: In Exercises 1 through 4, show that the given number $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ by finding $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$.

page 272, #1: $1 + \sqrt{2}$

$$\begin{aligned} \alpha &= 1 + \sqrt{2} \\ \alpha - 1 &= \sqrt{2} \\ \alpha^2 - 2\alpha + 1 &= 2 \\ \alpha^2 - 2\alpha - 1 &= 0 \end{aligned}$$

Let $f(x) = x^2 - 2x - 1$. Then $f(\alpha) = 0$.

page 272, #2: $\sqrt{2} + \sqrt{3}$

$$\begin{aligned} \alpha &= \sqrt{2} + \sqrt{3} \\ \alpha^2 &= 2 + 2\sqrt{6} + 3 \\ &= 5 + 2\sqrt{6} \\ \alpha^2 - 5 &= 2\sqrt{6} \\ \alpha^4 - 10\alpha^2 + 25 &= 24 \\ \alpha^4 - 10\alpha^2 + 1 &= 0 \end{aligned}$$

Let $f(x) = x^4 - 10x^2 + 1$. Then $f(\alpha) = 0$.

page 272, #3: $1 + i$

$$\begin{aligned} \alpha &= 1 + i \\ \alpha - 1 &= i \\ \alpha^2 - 2\alpha + 1 &= -1 \\ \alpha^2 - 2\alpha + 2 &= 0 \end{aligned}$$

Let $f(x) = x^2 - 2x + 2$. Then $f(\alpha) = 0$.

page 272, #4: $\sqrt{1 + \sqrt[3]{2}}$

$$\alpha = \sqrt{1 + \sqrt[3]{2}}$$

$$\alpha^2 = 1 + \sqrt[3]{2}$$
$$\alpha^2 - 1 = \sqrt[3]{2}$$
$$\alpha^6 - 3\alpha^4 + 3\alpha^2 - 1 = 2$$
$$\alpha^6 - 3\alpha^4 + 3\alpha^2 - 3 = 0$$

Let $f(x) = x^6 - 3x^4 + 3x^2 - 3$. Then $f(\alpha) = 0$.

page 272, #9, 10: In Exercises 9 and 10, classify the given $\alpha \in \mathbb{C}$ as algebraic or transcendental over the given field F. If α is algebraic over F, find $\deg(\alpha, F)$.

page 272, #9: $\alpha = i$, $F = \mathbb{Q}$

Since i is a root of $x^2 + 1$, it is algebraic over $\mathbb{Q}$.

Since $i \notin \mathbb{Q}$, i is not the root of any linear polynomial in $\mathbb{Q}[x]$, and $x^2 + 1$ is the minimal polynomial for i over $\mathbb{Q}$. Therefore $\deg(i, \mathbb{Q}) = 2$.

page 272, #10: $\alpha = 1 + i$, $F = \mathbb{R}$.

As seen in problem #3, $1 + i$ is a root of $x^2 - 2x + 2$. Since $1 + i \notin \mathbb{R}$, $x^2 - 2x + 2$ is a minimal polynomial for $1 + i$ over $\mathbb{R}$. Therefore $\deg(1 + i, \mathbb{R}) = 2$.

page 280, #1: Find three bases for $\mathbb{R}^2$ over $\mathbb{R}$, no two of which have a vector in common.

We need three disjoint sets, each consisting of a pair of linearly independent vectors. Here are three possible bases:

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right\}$$
$$\left\{\begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}\right\}$$

page 280, #2: Determine whether the given set of vectors is a basis for $\mathbb{R}^3$ over $\mathbb{R}$.

$$\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

We will form a linear combination and try to set the result equal to the zero vector.

$$a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) = (0, 0, 0)$$
$$(a + b, a + c, b + c) = (0, 0, 0)$$
$$a + b = 0$$
$$b = -a$$
$$a + c = 0$$
$$c = -a$$
$$b + c = 0$$
$$-2a = 0$$
$$a = 0$$
$$b = 0$$
$$c = 0$$

and the three vectors are linearly independent. Therefore they must form a basis for $\mathbb{R}^3$, which is three-dimensional.

page 291, #1, 2: In Exercises 1 through 2, find the degree and a basis for the given field extension. Be prepared to justify your answers.

page 291, #1: $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.

Since the minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is $x^2 - 2$, the degree is 2. The simplest basis would be $\{1, \sqrt{2}\}$.

page 291, #2: $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$.

$$\begin{aligned}[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] &= [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] \\ &= 2 \cdot 2 \\ &= 4\end{aligned}$$

One possible basis is $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$, which is the set of products from the basis $\{1, \sqrt{3}\}$ for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{2})$, and the basis $\{1, \sqrt{2}\}$ for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$.