Final Exam Solutions

Math 312-A

Friday, May 11, 2007

1. Define $\phi((m,n)) = m$. ϕ is a ring homomorphism since it preserves sums and products:

$$\begin{aligned}
\phi((m_1,n_1)+(m_2,n_2)) &= \phi((m_1+m_2,n_1+n_2)) \\
&= m_1+m_2 \\
&= \phi((m_1,n_1))+\phi((m_2,n_2)) \\
\phi((m_1,n_1)(m_2,n_2)) &= \phi((m_1m_2,n_1n_2)) \\
&= m_1m_2 \\
&= \phi((m_1,n_1))\,\phi((m_2,n_2))
\end{aligned}$$

   The kernel of ϕ is the subset of $\mathbb{Z}\times\mathbb{Z}$ where the first element is 0, which is the ideal $\langle(0,1)\rangle$.

   ϕ is onto, since for any $m\in\mathbb{Z}$, $m=\phi((m,0))$.

   By the Fundamental Homomorphism Theorem for Rings, $(\mathbb{Z}\times\mathbb{Z})/\langle(0,1)\rangle$ is isomorphic to $\mathbb{Z}$.

2.

$$\begin{aligned}
He &= H(1,4)(2,3) \\
&= H \\
&= \{e,(1,4)(2,3)\} \\
H(1,3,4,2) &= H(1,2,4,3) \\
&= \{(1,3,4,2),(1,2,4,3)\} \\
H(1,2)(3,4) &= H(1,3)(2,4) \\
&= \{(1,2)(3,4),(1,3)(2,4)\} \\
H(2,3) &= H(1,4) \\
&= \{(2,3),(1,4)\}
\end{aligned}$$

3. Since $45 = 3^2 5$, we are guaranteed subgroups of size 1, 3, 9, and 5.

   The Sylow 3-groups have size 9, and the number of them divides 45 and is congruent to 1 modulo 3. The only possible number is 1.

   The Sylow 5-groups have size 5, and the number of them divides 45 and is congruent to 1 modulo 5. The only possible number is 1.

4. Since $\sqrt[5]{3}$ has as its minimal polynomial x^5-3, the usual basis is $\{1, \sqrt[5]{3}, \sqrt[5]{3}^2, \sqrt[5]{3}^3, \sqrt[5]{3}^4\}$.

5. Since the dimension of $\mathbb{C}$ over $\mathbb{R}$ is finite (two, with basis $\{1,i\}$), no element of $\mathbb{C}$ can be transcendental over $\mathbb{R}$. Otherwise the powers of that element would be linearly independent and the dimension of $\mathbb{C}$ over $\mathbb{R}$ would be infinite.

6. Since $\langle 1,0,0\rangle\times\langle 1,0,0\rangle=\langle 0,0,0\rangle$, $\mathbb{R}^3$ has zero divisors.

7.

$$\begin{aligned}
e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\
e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) \\
&= \cos(\theta) - i\sin(\theta) \\
e^{i\theta}e^{-i\theta} &= (\cos(\theta) + i\sin(\theta))\,(\cos(\theta) - i\sin(\theta)) \\
1 &= \cos^2(\theta) - i^2\sin^2(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta)
\end{aligned}$$

8. Since in a finite field, adding 1 to itself must eventually repeat itself, we have for some distinct $m, n \in \mathbb{N}$ that $m \cdot 1 = n \cdot 1$. Assume without loss of generality that $m < n$. Then $(n - m) \cdot 1 = 0$ and the characteristic of the field is non-zero.

   If the characteristic were a compound number mn, then $(m \cdot 1)(n \cdot 1) = (mn) \cdot 1 = 0$, and we would have zero divisors, contradicting the fact that a field is an integer domain.

   Therefore the characteristic must be a prime number.

9. The nth roots of unity S are a subset of the multiplicative group of non-zero complex numbers. We need to show that S contains 1, inverses, and products.

   $1^n = 1$ so $1 \in S$.

   If $x \in S$, $\left(x^{-1}\right)^n = x^{-n} = (x^n)^{-1} = 1^{-1} = 1$, and $x^{-1} \in S$.

   If $x, y \in S$, then $(xy)^n = x^n y^n = 1 \cdot 1 = 1$, and $xy \in S$.

   Therefore, S is a multiplicative group.

10. One example would be the cyclic group generated by rotations of 1 radian. No integer multiple will ever be the identity, since no integer number of radians is coterminal with 0. Every non-zero element is of infinite order, and the group is infinite.

11. Let $\alpha = \sqrt{2} - \sqrt{3}$.

$$\begin{aligned}
\alpha &= \sqrt{2} - \sqrt{3} \\
\alpha^2 &= 2 - 2\sqrt{6} + 3 \\
&= 5 - 2\sqrt{6} \\
\alpha^2 - 5 &= -2\sqrt{6} \\
\left(\alpha^2 - 5\right)^2 &= 24 \\
\alpha^4 - 10\alpha^2 + 25 &= 24 \\
\alpha^4 - 10\alpha^2 + 1 &= 0
\end{aligned}$$

   and the minimal polynomial for α is $x^4 - 10x^2 + 1$.

12. Suppose that $p(x), q(x) \in \mathbb{Q}[x]$, and that $p(x)q(x) \in S$. Then $p(x)q(x)$ has 0 for a constant term, and therefore is a multiple of x. Either x divides $p(x)$ or x divides $q(x)$, i.e., either $p(x) \in S$ or $q(x) \in S$, and S is by definition a prime ideal.

13. Let $x, y, z \in \mathbb{R}$.

$$\begin{aligned}
f(f(x,y),z) &= f(x+y-xy,z)\\
&= x+y-xy+z-(x+y-xy)z\\
&= x+y+z-xy-xz-yz+xyz\\
f(x,f(y,z)) &= f(x,y+z-yz)\\
&= x+y+z-yz-x(y+z-yz)\\
&= x+y+z-xy-xz-yz+xyz\\
&= f(f(x,y),z)
\end{aligned}$$

and f is an associative binary operation.

14. Let e be the identity element that we are trying to find.

$$\begin{aligned}
f(x,e) &= x+e-xe\\
&= x\\
e-xe &= 0\\
e(1-x) &= 0 \text{ for all } x\\
e &= 0\\
f(0,x) &= 0+x-0\cdot x\\
&= x
\end{aligned}$$

and 0 is the identity element.

15. Let $(1,2,3) \in S_3$ and $(1,4) \in S_4$. Then $(1,4)(1,2,3)(1,4)^{-1} = (1,4)(1,2,3)(1,4) = (2,3,4) \notin S_3$. Since S_3 is not closed under conjugation by elements of S_4, S_3 is not a normal subgroup of S_4.