Homework #10 Solutions

Math 321-A

Due Friday, April 13, 2007

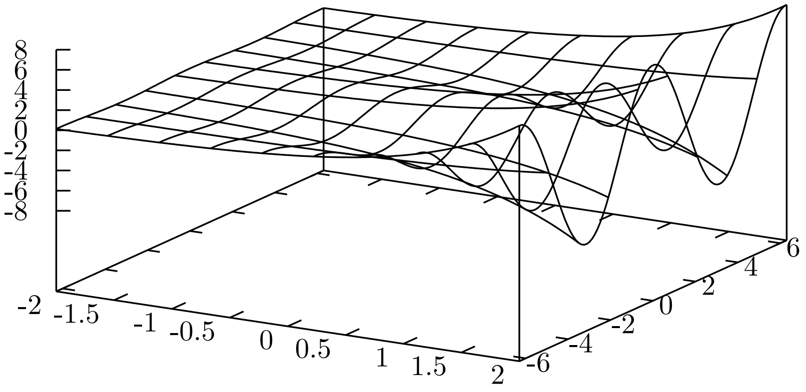
page 997, #13–16:

Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

page 997, #13:

$$\begin{aligned} f(x,y) &= e^x \cos(y) \\ \nabla f &= \langle e^x \cos(y), -e^x \sin(y) \rangle \end{aligned}$$

Since $e^x > 0$, and we can't have $\cos(y) = \sin(y) = 0$ by the Pythagorean Theorem, there are no critical points and thus no extrema.



page 997, #14:

Note that the domain of the functions excludes the x-axis and the y-axis.

$$\begin{aligned} f(x,y) &= x^2 + y^2 + \frac{1}{x^2y^2} \\ \nabla f &= \left\langle 2x - 2x^{-3}y^{-2}, 2y - 2x^{-2}y^{-3} \right\rangle \\ 2x - 2x^{-3}y^{-2} &= 0 \\ &= 2x^{-3}y^{-2}\left(x^4y^2 - 1\right) \\ x^4y^2 - 1 &= 0 \\ x^4y^2 &= 1 \\ y^2 &= x^{-4} \end{aligned}$$

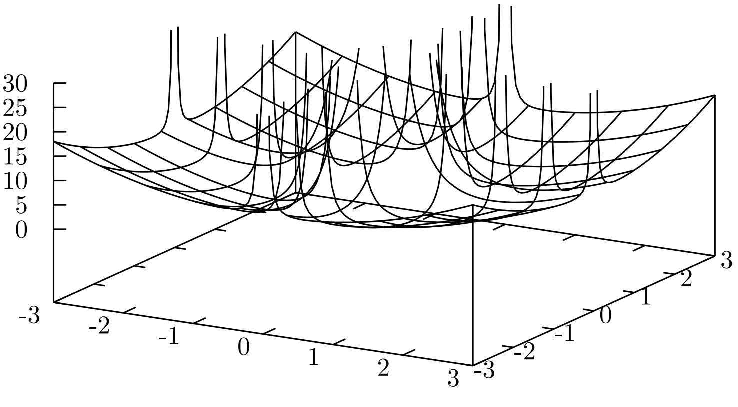
$$\begin{aligned}
y &= \pm x^{-2} \\
2y - 2x^{-2}y^{-3} &= 0 \\
2x^{-2}y^{-3}\left(x^2y^4 - 1\right) &= 0 \\
x^2y^4 - 1 &= 0 \\
x^2\left(x^{-8}\right) - 1 &= 0 \\
x^{-6} - 1 &= 0 \\
x^{-6} &= 1 \\
x^6 &= 1 \\
x^2 &= 1 \\
x &= \pm 1 \\
y &= \pm 1
\end{aligned}$$

There are four critical points in the domain: $(\pm 1, \pm 1)$.

$$\begin{aligned}
f_{xx} &= 2 + 6x^{-4}y^{-2} \\
f_{xy} &= 4x^{-3}y^{-3} \\
f_{yy} &= 2 + 6x^{-2}y^{-4}
\end{aligned}$$

(a, b)	$f_{xx}(a, b)$	$f_{yy}(a, b)$	$f_{xy}(a, b)$	D
$(-1, -1)$	8	8	4	$(8)(8) - 4^2 = 48 > 0$
$(-1, 1)$	8	8	-4	$(8)(8) - (-4)^2 = 48 > 0$
$(1, -1)$	8	8	-4	$(8)(8) - (-4)^2 = 48 > 0$
$(1, 1)$	8	8	4	$(8)(8) - 4^2 = 48 > 0$

By the Second Derivative Test, all four points are local minima, all with value $f(\pm 1, \pm 1) = 3$.



page 997, #15:

Here n will refer to an unknown integer.

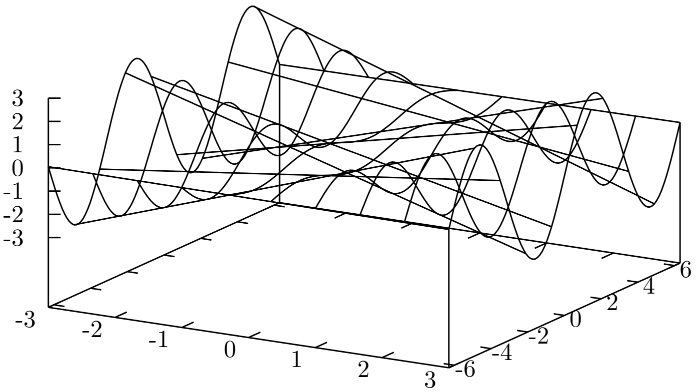
$$\begin{aligned}
f(x, y) &= x\sin(y) \\
\nabla f &= \langle \sin(y), x\cos(y) \rangle
\end{aligned}$$

$$
\begin{aligned}
\sin(y) &= 0 \\
y &= n\pi \\
x\cos(n\pi) &= 0 \\
x(-1)^n &= 0 \\
x &= 0
\end{aligned}
$$

The function has critical points $(0, n\pi)$ for all integers n.

$$
\begin{aligned}
f_{xx} &= 0 \\
f_{yy} &= -x\sin(y) \\
f_{xy} &= \cos(y) \\
D(0, n\pi) &= (0)(0) - (\cos(n\pi))^2 \\
&= -1 \\
&< 0
\end{aligned}
$$

All of the critical points are saddle points. $f(0, n\pi) = 0$.



page 997, #16:

$$
\begin{aligned}
f(x, y) &= \left(2x - x^2\right)\left(2y - y^2\right) \\
\nabla f &= \left\langle (2-2x)\left(2y - y^2\right), \left(2x - x^2\right)(2-2y) \right\rangle \\
(2-2x)\left(2y - y^2\right) &= 0 \\
2(1-x)y(2-y) &= 0 \\
\left(2x - x^2\right)(2-2y) &= 0 \\
x(2-x)(2)(1-y) &= 0
\end{aligned}
$$

Looking at $f_x = 0$, either $x = 1$ or $y = 0$ or $y = 2$.

Looking at $f_y = 0$, either $x = 0$ or $x = 2$ or $y = 1$.

If $x = 1$, then $y = 1$.

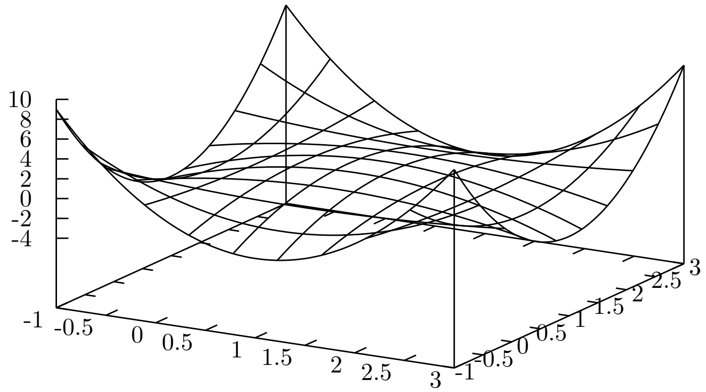
If $x \neq 1$, then $y = 0$ or $y = 2$. If $x \neq 1$ and $y = 0$, then $x = 0$ of $x = 2$. If $x \neq 1$ and $y = 2$, then $x = 0$ or $x = 2$.

There are five critical points: $(1,1)$, $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$.

$$\begin{aligned} f_{xx} &= -2\left(2y - y^2\right) \\ f_{yy} &= -2\left(2x - x^2\right) \\ f_{xy} &= (2-2x)(2-2y) \end{aligned}$$

(a,b)	$f_{xx}(a,b)$	$f_{yy}(a,b)$	$f_{xy}(a,b)$	$D(a,b)$
$(1,1)$	-2	-2	0	$(-2)(-2) - 0^2 = 4 > 0$
$(0,0)$	0	0	4	$(0)(0) - 4^2 = -16 < 0$
$(0,2)$	0	0	-4	$(0)(0) - (-4)^2 = -16 < 0$
$(2,0)$	0	0	-4	$(0)(0) - (-4)^2 = -16 < 0$
$(2,2)$	0	0	4	$(0)(0) - 4^2 = -16 < 0$

and by the Second Derivative Test, $(1,1)$ is a local maximum, while the other four points are saddle points. Note that $f(1,1) = 1$ while $f = 0$ at the other four critical points.



page 997, #29, 30:

Find the absolute maximum and minimum values of f on the set D.

page 997, #29:

$$\begin{aligned} f(x,y) &= x^2 + y^2 + x^2 y + 4 \\ \nabla f &= \left\langle 2x + 2xy, 2y + x^2 \right\rangle \\ 2x + 2xy &= 0 \\ 2x(1+y) &= 0 \\ 2y + x^2 &= 0 \end{aligned}$$

If $x = 0$, then $y = 0$. If $x \neq 0$, $y = -1$ and $x = \pm\sqrt{2}$. There is only one critical point inside D, at $(0,0)$.

The boundary of D are the four line segments:

(a) $x = -1, -1 \leq y \leq 1$
$f(-1,y) = y^2 + y + 5$, which has a minimum at $y = -1/2$ where $f(-1,-1/2) = 4.75$.

(b) $x = 1, -1 \leq y \leq 1$
$f(1, y) = y^2 + y + 5$ which has a minimum at $y = -1/2$ where $f(1, -1/2) = 4.75$.

(c) $y = -1, -1 \leq x \leq 1$
$f(x, -1) = 5$ which is a constant.

(d) $y = 1, -1 \leq x \leq 1$
$f(x, 1) = 2x^2 + 5$, which takes a minimum at $x = 0$ where $f(0, 1) = 5$.

The boundaries of the four segments are at the corners $f(-1, -1) = 5$, $f(-1, 1) = 7$, $f(1, -1) = 5$, and $f(1, 1) = 7$.

(a, b)	$f(a, b)$	Conclusion
$(0, 0)$	4	absolute minimum
$(-1, -1/2)$	4.75	
$(1, -1/2)$	4.75	
$(x, -1), -1 \leq x \leq 1$	5	
$(0, 1)$	5	
$(-1, -1)$	5	
$(-1, 1)$	7	absolute maximum
$(1, -1)$	5	
$(1, 1)$	7	absolute maximum

page 997, #30:

$$\begin{aligned} f(x, y) &= 4x + 6y - x^2 - y^2 \\ \nabla f &= \langle 4 - 2x, 6 - 2y \rangle \\ 4 - 2x &= 0 \\ x &= 2 \\ 6 - 2y &= 0 \\ y &= 3 \end{aligned}$$

and there is one interior critical point, $(2, 3)$.

The boundary of D are the four line segments:

(a) $x = 0, 0 \leq y \leq 5$
$f(0, y) = 6y - y^2$, which has a maximum at $y = 3$ where $f(0, 3) = 9$.

(b) $x = 4, 0 \leq y \leq 5$
$f(4, y) = 6y - y^2$ which has a maximum at $y = 3$ where $f(4, 3) = 9$.

(c) $y = 0, 0 \leq x \leq 4$
$f(x, 0) = 4x - x^2$ which has a maximum at $x = 2$ where $f(2, 0) = 4$.

(d) $y = 5, 0 \leq x \leq 4$
$f(x, 5) = 4x - x^2 + 5$, which takes a maximum at $x = 2$ where $f(2, 5) = 9$.

The boundaries of the four segments are at the corners $f(0, 0) = 0$, $f(0, 5) = 5$, $f(4, 0) = 0$, and $f(4, 5) = 5$.

(a,b)	$f(a,b)$	Conclusion
$(2,3)$	13	absolute maximum
$(0,3)$	9	
$(4,3)$	9	
$(2,0)$	4	
$(2,5)$	9	
$(0,0)$	0	absolute minimum
$(0,5)$	5	
$(4,0)$	0	absolute minimum
$(4,5)$	5	

page 997, #39:

Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

$$\begin{aligned}
\text{distance}^2 &= x^2 + y^2 + z^2 \\
&= x^2 + y^2 + xy + 1 \\
&= f(x,y) \\
\nabla f &= \langle 2x + y, 2y + x \rangle \\
2x + y &= 0 \\
y &= -2x \\
2y + x &= 0 \\
-4x + x &= 0 \\
-3x &= 0 \\
x &= 0 \\
y &= 0 \\
z^2 &= (0)(0) + 1 \\
&= 1
\end{aligned}$$

and the closest points are $(0, 0, \pm 1)$.

page 997, #40:

Find the points on the surface $x^2y^2z = 1$ that are closest to the origin.

Note that none of x, y, or z can be 0.

$$\begin{aligned}
x^2y^2z &= 1 \\
z &= x^{-2}y^{-2} \\
\text{distance}^2 &= x^2 + y^2 + z^2 \\
&= x^2 + y^2 + x^{-4}y^{-4} \\
&= f(x,y) \\
\nabla f &= \left\langle 2x - 4x^{-5}y^{-4}, 2y - 4x^{-4}y^{-5} \right\rangle \\
2x - 4x^{-5}y^{-4} &= 0 \\
2x^{-5}y^{-4}\left(x^6y^4 - 2\right) &= 0 \\
x^6y^4 &= 2 \\
2y - 4x^{-4}y^{-5} &= 0
\end{aligned}$$

$$\begin{aligned}
2x^{-4}y^{-5}\left(x^4y^6-2\right) &= 0\\
x^4y^6 &= 2\\
\frac{x^6y^4}{x^4y^6} &= \frac{2}{2}\\
\frac{x^2}{y^2} &= 1\\
x^2 &= y^2\\
x^{10} &= 2\\
x &= \pm\sqrt[10]{2}\\
y &= \pm\sqrt[10]{2}\\
z &= 2^{-2/5}
\end{aligned}$$

and there are four points: $\left(\pm\sqrt[10]{2}, \pm\sqrt[10]{2}, 2^{-2/5}\right)$.

page 1024, #3:

(a) Use a Riemann sum with $m = n = 2$ to estimate the value of $\iint_R \sin(x+y)\,dA$, where $R = [0,\pi]\times[0,\pi]$. Take the sample points to be the lower left corners.

$$\begin{aligned}
\sin(0+0)\left(\frac{\pi}{2}\right)^2 + \sin\left(0+\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)^2 + \sin\left(\frac{\pi}{2}+0\right)\left(\frac{\pi}{2}\right)^2 + \sin\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)^2 &= (0+1+1+0)\left(\frac{\pi^2}{4}\right)\\
&= \frac{\pi^2}{2}\\
&= 4.935
\end{aligned}$$

(b) Use the Midpoint Rule to estimate the integral in part (a).

$$\begin{aligned}
\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right)^2 + \sin\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)\left(\frac{\pi}{2}\right)^2 + \sin\left(\frac{3\pi}{4}+\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right)^2 + \sin\left(\frac{3\pi}{4}+\frac{3\pi}{4}\right)\left(\frac{\pi}{2}\right)^2 &= (1+0+0+-1)\left(\frac{\pi^2}{4}\right)\\
&= 0
\end{aligned}$$

page 1024, #4:

(a) Estimate the volume of the solid that lies below the surface $z = x + 2y^2$ and above the rectangle $R = [0,2]\times[0,4]$. Use a Riemann sum with $m = n = 2$ and choose the sample points to be lower right corners.

$$\begin{aligned}
\left(1+2(0)^2\right)(1\cdot 2) + \left(2+2(0)^2\right)(1\cdot 2) + \left(1+2(2)^2\right)(1\cdot 2) + \left(2+2(2)^2\right)(1\cdot 2) &= (1+2+9+10)(2)\\
&= 44
\end{aligned}$$

(b) Use the Midpoint Rule to estimate the volume in part (a).

$$\begin{aligned}
\left(0.5+2(1)^2\right)(1\cdot 2) + \left(1.5+2(1)^2\right)(1\cdot 2) + \left(0.5+2(3)^2\right)(1\cdot 2) + \left(1.5+2(3)^2\right)(1\cdot 2) &= (2.5+3.5+18.5+19.5)(2)\\
&= 88
\end{aligned}$$