Exam #1 Solutions

Math 321-A

Thursday, March 1, 2007

1.

$$
\begin{aligned}
5x - 2y - 3z &= 8 \\
5(3 - t) - 2(4 - 2t) - 3(5 + 4t) &= 8 \\
15 - 5t - 8 + 4t - 15 - 12t &= 8 \\
-13t &= 16 \\
t &= -\frac{16}{13} \\
x &= 3 - t \\
&= 3 - \left(-\frac{16}{13}\right) \\
&= \frac{55}{13} \\
&= 4.231 \\
y &= 4 - 2t \\
&= 4 - 2\left(-\frac{16}{13}\right) \\
&= \frac{84}{13} \\
&= 6.462 \\
z &= 5 + 4\left(-\frac{16}{13}\right) \\
&= \frac{1}{13} \\
&= 0.077
\end{aligned}
$$

2. The hyperbola

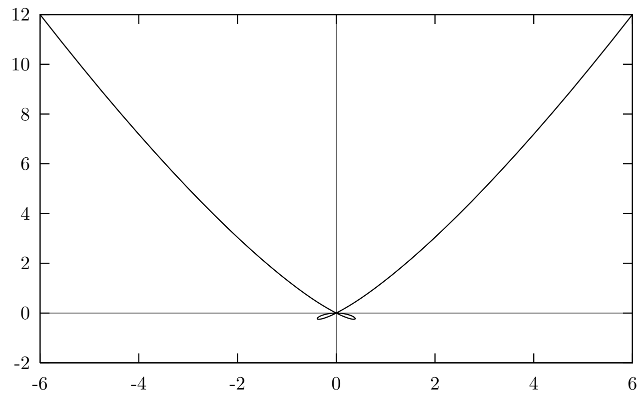
$$x^2 - y^2 = 1$$

opens to the left and right. Rotating it around the y-axis joins these two branches, forming a hyperboloid of one sheet.

3.

$$
\begin{aligned}
(x + 4)^2 + y^2 &= 16 \\
(r\cos(\theta) + 4)^2 + (r\sin(\theta))^2 &= 16 \\
r^2\cos^2(\theta) + 8r\cos(\theta) + 16 + r^2\sin^2(\theta) &= 16 \\
r^2\left(\cos^2(\theta) + \sin^2(\theta)\right) + 8r\cos(\theta) &= 0 \\
r^2 + 8r\cos(\theta) &= 0
\end{aligned}
$$

4.



The point of self-intersection is at the origin.

$$\begin{aligned} x &= t^3 - t \\ &= t(t-1)(t+1) \\ y &= t^4 - t^2 \\ &= t^2(t-1)(t+1) \end{aligned}$$

and the origin occurs at $t = -1, 0, 1$.

5. We will solve for when the slope is undefined.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{4t^3 - 2t}{3t^2 - 1} \\ 3t^2 - 1 &= 0 \\ 3t^2 &= 1 \\ t^2 &= \frac{1}{3} \\ t &= \pm\frac{1}{\sqrt{3}} \\ &= \pm 0.577 \end{aligned}$$

6. The angle formed by the planes is congruent to the angles formed by their respective normal vectors, and we will measure that.

$$\begin{aligned} 5x - y + 2z &= 11 \\ \overrightarrow{N_1} &= \langle 5, -1, 2 \rangle \\ 3x + 9z &= 12 \\ \overrightarrow{N_2} &= \langle 3, 0, 9 \rangle \\ \cos(\theta) &= \frac{\overrightarrow{N_1} \cdot \overrightarrow{N_2}}{\left\|\overrightarrow{N_1}\right\| \left\|\overrightarrow{N_2}\right\|} \end{aligned}$$

$$\begin{aligned} &= \frac{15+0+18}{\sqrt{25+1+4}\sqrt{9+0+81}} \\ &= \frac{33}{\sqrt{30}\sqrt{90}} \\ &= \frac{11}{10\sqrt{3}} \\ \theta &= 0.883 \end{aligned}$$

7. We first connect the point $Q = (3, 1, -5)$ and the line $(x, y, z) = (2-t, 4+3t, 2t-4) = P + t\overrightarrow{A}$ where $P = (2, 4, -4)$ and $\overrightarrow{A} = \langle -1, 3, 2\rangle$. We'll let $\overrightarrow{PQ} = Q - P = \langle 1, -3, -1\rangle$.

   Next we'll compute the perpendicular component of $\overrightarrow{PQ}$. We'll first find the parallel component as the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{A}$, and then subtract it from $\overrightarrow{PQ}$.

$$\begin{aligned} \text{proj}_{\overrightarrow{A}}\,\overrightarrow{PQ} &= \frac{\overrightarrow{PQ}\cdot\overrightarrow{A}}{\overrightarrow{A}\cdot\overrightarrow{A}}\overrightarrow{A} \\ &= \frac{-1-9-2}{1+9+4}\langle -1, 3, 2\rangle \\ &= -\frac{6}{7}\langle -1, 3, 2\rangle \\ &= \left\langle \frac{6}{7}, -\frac{18}{7}, -\frac{12}{7}\right\rangle \\ \overrightarrow{PQ} - \text{proj}_{\overrightarrow{A}}\,\overrightarrow{PQ} &= \langle 1, -3, -1\rangle - \left\langle \frac{6}{7}, -\frac{18}{7}, -\frac{12}{7}\right\rangle \\ &= \left\langle \frac{1}{7}, -\frac{3}{7}, \frac{5}{7}\right\rangle \\ &= \frac{1}{7}\langle 1, -3, 5\rangle \end{aligned}$$

   The perpendicular distance is the length of this vector.

$$\begin{aligned} \left\|\frac{1}{7}\langle 1, -3, 5\rangle\right\| &= \frac{1}{7}\sqrt{1+9+25} \\ &= \frac{1}{7}\sqrt{35} \\ &= 0.845 \end{aligned}$$

8. We divide out the length of the vector to create a unit vector in the same direction.

$$\begin{aligned} \frac{\langle 2, 5, 7\rangle}{\|\langle 2, 5, 7\rangle\|} &= \frac{1}{\sqrt{4+25+49}}\langle 2, 5, 7\rangle \\ &= \left\langle \frac{2}{\sqrt{78}}, \frac{5}{\sqrt{78}}, \frac{7}{\sqrt{78}}\right\rangle \\ &= \langle 0.226, 0.566, 0.793\rangle \end{aligned}$$

9. The cross product of two vectors is always perpendicular to the two vectors, which requires a third dimension.

10. The standard equation for $y = x^2 - 20$ is $4p(y-k) = (x-h)^2$:

$$y = x^2 - 20$$

$$y + 20 = x^2$$
$$4\left(\frac{1}{4}\right)(y - (-20)) = (x - 0)^2$$

The center is $(h, k) = (0, -20)$ and $p = 1/4$. The parabola opens upward, and the focus is $1/4$ above $(0, -20)$, i.e., $(0, -79/4) = (0, -19.75)$.