Exam #3 Solutions

Math 321-A

Thursday, May 3, 2007

1. Here are some possible answers:
   - Need to use partial derivatives instead of total derivatives.
   - There's no first derivative test.
   - Saddle points are possible.
   - The second derivative test involves several things to check.

2. Note that $2x + 3y > 0$ on our domain.

$$\begin{aligned}
f(x,y) &= \ln(2x+3y) \\
f_x &= \frac{2}{2x+3y} \\
f_y &= \frac{3}{2x+3y} \\
\sqrt{f_x^2 + f_y^2 + 1} &= \sqrt{\left(\frac{2}{2x+3y}\right)^2 + \left(\frac{3}{2x+3y}\right)^2 + 1} \\
&= \frac{1}{2x+3y}\sqrt{4+9+(2x+3y)^2} \\
&= \frac{\sqrt{13+4x^2+12xy+9y^2}}{2x+3y} \\
\text{Surface Area} &= \int_2^4 \int_1^x \frac{\sqrt{13+4x^2+12xy+9y^2}}{2x+3y}\,dy\,dx
\end{aligned}$$

3.

$$\begin{aligned}
\iint\limits_R (5x+6y)\,dA &= \int_0^\pi \int_0^3 \Big(5r\cos(\theta) + 6\sin(\theta)\Big) r\,dr\,d\theta \\
&= \int_0^\pi \int_0^3 \Big(5\cos(\theta) + 6\sin(\theta)\Big) r^2\,dr\,d\theta \\
&= \int_0^\pi \Big(5\cos(\theta) + 6\sin(\theta)\Big) \frac{r^3}{3}\Big|_{r=0}^{r=3}\,d\theta \\
&= \int_0^\pi \Big(5\cos(\theta) + 6\sin(\theta)\Big)(9-0)\,d\theta \\
&= 9\Big(5\sin(\theta) - 6\cos(\theta)\Big)\Big|_{\theta=0}^{\theta=\pi} \\
&= 9\Big(0 - 6(-1)\Big) - 9\Big(0 - 6(1)\Big) \\
&= 108
\end{aligned}$$

4.

$$f(x,y) = x^3 - 24xy + y^3$$
$$f_x = 3x^2 - 24y$$
$$f_y = -24x + 3y^2$$

First we solve for all critical points.

$$3x^2 - 24y = 0$$
$$3x^2 = 24y$$
$$y = \frac{x^2}{8}$$
$$-24x + 3y^2 = 0$$
$$-24x + 3\left(\frac{x^2}{8}\right)^2 = 0$$
$$-24x + \frac{3x^4}{64} = 0$$
$$-8x + \frac{x^4}{64} = 0$$
$$-512x + x^4 = 0$$
$$x\left(-512 + x^3\right) = 0$$
$$x = 0, 8$$

When $x = 0$, $y = 0$. When $x = 8$, $y = 8$. $f(0,0) = 0$ and $f(8,8) = -512$.

Now we use the Second Partials Test to sort the points.

$$f_{xx} = 6x$$
$$f_{xy} = -24$$
$$f_{yy} = 6y$$

Point	f_{xx}	f_{yy}	f_{xy}	D	Conclusion
$(0,0)$	0	0	−24	−576	Saddle point
$(8,8)$	48	48	−24	1728	Local minimum

5. We need to find the points of intersection of the two curves.

$$x^2 + x + 1 = -2x^2 - 5x + 10$$
$$3x^2 + 6x - 9 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$

The intersections occur at $x = -3$ and $x = 1$. Looking at the x-intercepts, we see that $x^2 + x + 1$ is on the bottom and $-2x^2 - 5x + 10$ is on the top.

$$M_y = \iint_R x\,dA$$
$$= \int_{-3}^{1} \int_{x^2+x+1}^{-2x^2-5x+10} x\,dy\,dx$$

$$
\begin{aligned}
&= \int_{-3}^{1} xy\Big|_{y=x^2+x+1}^{y=-2x^2-5x+10} \, dx \\
&= \int_{-3}^{1} x\left(\left(-2x^2-5x+10\right)-\left(x^2+x+1\right)\right) dx \\
&= \int_{-3}^{1} x\left(-3x^2-6x+9\right) dx \\
&= \int_{-3}^{1} \left(-3x^3-6x^2+9x\right) dx \\
&= -\frac{3x^4}{4}-2x^3+\frac{9x^2}{2}\Big|_{x=-3}^{x=1} \\
&= \left(-\frac{3}{4}-2+\frac{9}{2}\right)-\left(-\frac{243}{4}+54+\frac{81}{2}\right) \\
&= \frac{-3-8+18+243-216-162}{4} \\
&= -\frac{128}{4} \\
&= -32 \\
m &= \iint_R dA \\
&= \int_{-3}^{1}\int_{x^2+x+1}^{-2x^2-5x+10} dy\, dx \\
&= \int_{-3}^{1} y\Big|_{y=x^2+x+1}^{y=-2x^2-5x+10} \, dx \\
&= \int_{-3}^{1} \left(\left(-2x^2-5x+10\right)-\left(x^2+x+1\right)\right) dx \\
&= \int_{-3}^{1} \left(-3x^2-6x+9\right) dx \\
&= -x^3-3x^2+9x\Big|_{x=-3}^{x=1} \\
&= (-1-3+9)-(27-27-27) \\
&= 32 \\
\overline{x} &= \frac{M_y}{m} \\
&= \frac{-32}{32} \\
&= -1
\end{aligned}
$$

6. We need to check for interior critical points and then for possible extrema on the boundary.

$$
\begin{aligned}
f(x,y) &= x^2y \\
\nabla f(x,y) &= \langle 2xy, x^2\rangle \\
x^2 &= 0 \\
x &= 0
\end{aligned}
$$

and $-1 < y < 1$. There are an infinite number of interior critical points.

The boundary consists of four sides.

- $x = 2$ and $-1 \le y \le 1$: $f(2, y) = 4y$ takes its minimum at $y = -1$ and its maximum at $y = 1$.
- $x = -2$ and $-1 \le y \le 1$: $f(-2, y) = 4y$ takes its minimum at $y = -1$ and its maximum at $y = 1$.
- $-2 \le x \le 2$ and $y = -1$: $f(x, -1) = -x^2$ takes its minimum at $x = \pm 2$ and its maximum at $x = 0$.
- $-2 \le x \le 2$ and $y = 1$: $f(x, 1) = x^2$ takes its minimum at $x = 0$ and its maximum at $x = \pm 2$.

(a, b)	$f(a, b)$	Conclusion
$(0, y)$ for $-1 < y < 1$	0	
$(\pm 2, -1)$	-4	Absolute minima
$(\pm 2, 1)$	4	Absolute maxima
$(0, -1)$	0	
$(0, 1)$	0	

7. We need to integrate the probability density function over the region $xy \le 3$ in the domain, i.e., where $y \le 3/x$ and $1 \le x \le 3$.

$$\begin{aligned}
\Pr(xy \le 3) &= \int_1^3 \int_1^{3/x} \frac{x+y}{16}\, dy\, dx \\
&= \frac{1}{16} \int_1^3 \left(xy + \frac{y^2}{2} \right) \Big|_{y=1}^{y=3/x} dx \\
&= \frac{1}{16} \int_1^3 \left(\left(3 + \frac{9x^{-2}}{2} \right) - \left(x + \frac{1}{2} \right) \right) dx \\
&= \frac{1}{16} \int_1^3 \left(\frac{5}{2} + \frac{9x^{-2}}{2} - x \right) dx \\
&= \frac{1}{16} \left(\frac{5x}{2} + \frac{9x^{-1}}{-2} - \frac{x^2}{2} \right) \Big|_{x=1}^{x=3} \\
&= \frac{1}{16} \left(\left(\frac{15}{2} - \frac{3}{2} - \frac{9}{2} \right) - \left(\frac{5}{2} - \frac{9}{2} - \frac{1}{2} \right) \right) \\
&= \frac{1}{16} \frac{8}{2} \\
&= \frac{1}{4} \\
&= 25\%
\end{aligned}$$

8. We partition the interval for x ($[0, 6]$) into two parts ($[0, 3]$ and $[3, 6]$) and likewise the interval for y ($[1, 5]$) into two parts ($[1, 3]$ and $[3, 5]$). The upper right-hand points are $(3, 3)$, $(3, 5)$, $(6, 3)$, and $(6, 5)$. We have that $\Delta x = 3$ and $\Delta y = 2$.

$$\begin{aligned}
\text{Riemann Sum} &= f(3,3)\Delta x \Delta y + f(3,5)\Delta x \Delta y + f(6,3)\Delta x \Delta y + f(6,5)\Delta x \Delta y \\
&= \left(3^2 + 3^3\right)(3)(2) + \left(3^2 + 5^3\right)(3)(2) + \left(6^2 + 3^3\right)(3)(2) + \left(6^2 + 5^3\right)(3)(2) \\
&= 216 + 804 + 378 + 966 \\
&= 2364
\end{aligned}$$

9.

$$\text{Volume} = \int_1^5 \int_1^{5/x} xy\, dy\, dx$$

$$
\begin{aligned}
&= \int_1^5 \frac{xy^2}{2}\bigg|_{y=1}^{y=5/x} dx \\
&= \frac{1}{2}\int_1^5 \left(25x^{-1} - x\right) dx \\
&= \frac{1}{2}\left(25\ln(x) - \frac{x^2}{2}\right)\bigg|_{x=1}^{x=5} \\
&= \frac{1}{2}\left(\left(25\ln(5) - \frac{25}{2}\right) - \left(0 - \frac{1}{2}\right)\right) \\
&= \frac{25\ln(5)}{2} - 6 \\
&= 14.118
\end{aligned}
$$

10. The region in cylindrical coordinates is $0 \le r \le 2$, $0 \le \theta \le 2\pi$, and $-3 \le z \le 5$.
    The integrand is $e^{x^2+y^2} = e^{r^2}$.
    The differential is $dV = r\,dr\,d\theta\,dz$.
    We will use the substitution $u = r^2$ and $du = 2r\,dr$.

$$
\begin{aligned}
\iiint_R e^{x^2+y^2}\,dV &= \int_{-3}^5 \int_0^{2\pi} \int_0^2 e^{r^2} r\,dr\,d\theta\,dz \\
&= \int_{-3}^5 \int_0^{2\pi} \int_0^4 e^u \frac{du}{2}\,d\theta\,dz \\
&= \frac{1}{2}\int_{-3}^5 \int_0^{2\pi} e^u\bigg|_{u=0}^{u=4} d\theta\,dz \\
&= \frac{e^4 - 1}{2}\int_{-3}^5 \int_0^{2\pi} d\theta\,dz \\
&= \left(e^4 - 1\right)\pi \int_{-3}^5 dz \\
&= 8\left(e^4 - 1\right)\pi \\
&= 1347.068
\end{aligned}
$$