Kepler's Laws and Newton's Law of Gravity

Math 321-A

Spring, 2008

Kepler's Laws

In the early seventeenth century, Kepler formulated three laws to describe the motions of the planets based on astronomical data collected by Brahe. Kepler's three laws describe the shape of the orbits as well as connections between their size and speed.

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1. The orbits of the planets around the sun are ellipses, with the Sun at one focus of the ellipse.
2. The line joining a planet to the Sun sweeps out equal areas in equal times as the planet travels around its orbit.
3. Let a_1 and a_2 be the semi-major axes for two planets traveling around the same sun, and let T_1 and T_2 be the length of their years, i.e., the time it takes for each planet to complete an orbit.

   $$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2}$$

   This is equivalent to saying that a^3/T^2 is a constant independent of which planet we are measuring.

Newton's Inverse Square Law

Thirty years later Newton formulated his inverse square law of gravity. This law is remarkable both for its simplicity as well as its ability to explain the patterns present in Kepler's laws. This presentation will show how to derive Kepler's Laws from Newton's inverse square law of gravity.

Statement of Newton's Law of Gravity

Newton's Law of Gravity states that the force of gravity between two objects is proportional to the mass of each of the objects and inversely proportional to the square of the distance between them. If we let G be a proportionality constant, then

$$F = \frac{GMm}{r^2}$$

where F is the magnitude of force between the two objects, M is the mass of one of the objects, m is the mass of the other object, and r is the distance between them. The force is directed from each object to the other.

Planetary Orbits

Let's let the first object be the Sun with mass M and the second object a planet with mass m. We will use Newton's Second Law of Motion $F = ma$, where a is the magnitude of acceleration, to find the magnitude of acceleration experienced by the planet.

$$\begin{aligned} a &= \frac{F}{m} \\ &= \frac{GM}{r^2} \end{aligned}$$

Choice of Origin

We will locate the Sun at the origin. Then the acceleration is directed at the origin. The gravity experienced by the planet is an example of a central force, one which is directed at a specific point in space.

Central Forces

We will show that planets under the influence of any central force must travel in planar orbits. Let $\vec{r}(t)$ be the position vector for the planet at time t. For a central force, the acceleration $\vec{a}(t)$ is parallel to $\vec{r}(t)$. We can show that the orbit is planar by creating a constant vector $\vec{N}$ such that $\vec{N} \cdot \vec{r}(t) = 0$ for all t; then the orbit is contained in the plane $\vec{N} \cdot (x, y, z) = 0$.

Showing That the Orbit is Planar

Let $\vec{N} = \vec{r}(t) \times \vec{v}(t)$, where $\vec{v}(t) = d\vec{r}(t)/dt$ is the velocity vector. We will also need $\vec{a}(t) = d\vec{v}(t)/dt$, the acceleration vector. Because the force is central, $\vec{a}(t) = K\vec{r}(t)$ for some constant K.

$$\frac{d\vec{N}}{dt} = \frac{d}{dt}\left(\vec{r}(t) \times \vec{v}(t)\right)$$

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$$\begin{aligned}\frac{d\vec{N}}{dt} &= \frac{d}{dt}\left(\vec{r}(t) \times \vec{v}(t)\right)\\ &= \frac{d\vec{r}(t)}{dt} \times \vec{v}(t) + \vec{r}(t) \times \frac{d\vec{v}(t)}{dt}\end{aligned}$$

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\frac{d\vec{N}}{dt} &= \frac{d}{dt}\left(\vec{r}(t) \times \vec{v}(t)\right) \\
&= \frac{d\vec{r}(t)}{dt} \times \vec{v}(t) + \vec{r}(t) \times \frac{d\vec{v}(t)}{dt} \\
&= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t)
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&= \frac{d\vec{r}(t)}{dt} \times \vec{v}(t) + \vec{r}(t) \times \frac{d\vec{v}(t)}{dt} \\
&= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t) \\
&= \vec{0} + \vec{r}(t) \times (K\vec{r}(t))
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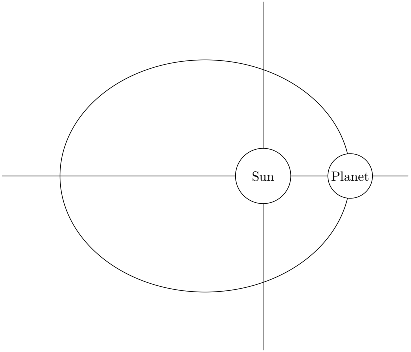
&= \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t) \\
&= \vec{0} + \vec{r}(t) \times (K\vec{r}(t)) \\
&= K\vec{r}(t) \times \vec{r}(t) \\
&= \vec{0}
\end{aligned}$$

Constant Normal Vector for Plane of Orbit

Therefore $\vec{N}$ is a constant vector. Since it is a cross product of $\vec{r}(t)$ with another vector, $\vec{N} \cdot \vec{r}(t) = 0$ for all t. Therefore the orbit lies in the plane $\vec{N} \cdot (x, y, z) = 0$.

Choosing Coordinate Axes



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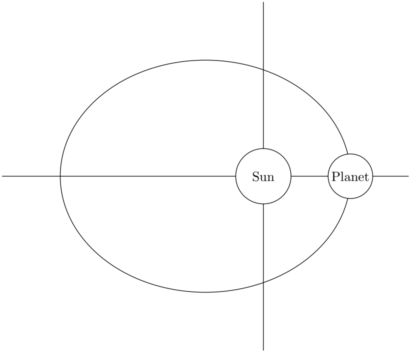

- Since the orbit is planar, with the plane of the orbit containing the Sun at the origin, we will focus on the *xy*-plane.
- For any orbit, there is always a position where the planet is closest to the Sun, known as the perihelion. We will draw the coordinate axes through the sun so that the perihelion lies on the positive *x*-axis.

Perihelion

The perihelion will be our reference point. We will call the time of perihelion t_0, the distance r_0, and in general whenever we use the subscript 0 we will refer to the perihelion. The angle θ_0 is 0. Note that since r is at a minimum at perihelion:

$$\left.\frac{dr}{dt}\right|_{t=t_0} = 0$$

We will also choose our orientation so that the planet travels counter-clockwise at perihelion:

$$\left.\frac{d\theta}{dt}\right|_{t=t_0} > 0 \tag{1}$$

Polar Unit Vectors

Since we will be working with vectors and polar coordinates, we will define a set of unit vectors that describe how the radius is changing and how the angle is changing.
Since $(x, y) = (r\cos(\theta), r\sin(\theta)) = r(\cos(\theta), \sin(\theta))$, we start by defining

$$\vec{u}_r = (\cos(\theta), \sin(\theta))$$

Note that by the Pythagorean Identity, $\vec{u}_r$ has length 1.

Polar Unit Vectors (continued)

We also want a perpendicular unit vector that indicates how the angle to the planet is changing. The easiest way is to rotate $\vec{u}_r$ by a right angle counter-clockwise:

$$\vec{u}_\theta = \left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right)$$

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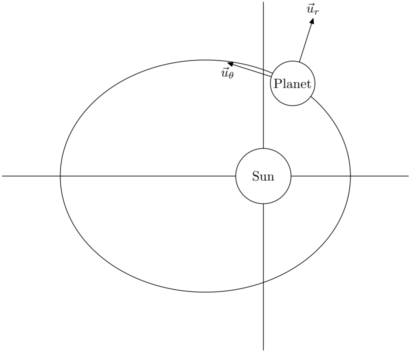
&\qquad \left.\sin(\theta)\cos\left(\frac{\pi}{2}\right) + \cos(\theta)\sin\left(\frac{\pi}{2}\right)\right) \\
&= (-\sin(\theta), \cos(\theta))
\end{aligned}$$

Polar Unit Vectors (continued)



$\vec{u}_\theta$ is also a unit vector and it is perpendicular to $\vec{u}_r$.
Note also:

$$\begin{aligned}\frac{d\vec{u}_r}{d\theta} &= \frac{d}{d\theta}\left(\cos(\theta), \sin(\theta)\right)\\ &= (-\sin(\theta), \cos(\theta))\\ &= \vec{u}_\theta\\ \frac{d\vec{u}_\theta}{d\theta} &= \frac{d}{d\theta}\left(-\sin(\theta), \cos(\theta)\right)\\ &= (-\cos(\theta), -\sin(\theta))\\ &= -\vec{u}_r\end{aligned}$$

Polar Unit Vectors (continued)

Given $\vec{r} = a\vec{u}_r + b\vec{u}_\theta$ for any a and b, then we can solve for a and b:

$$\begin{aligned}
\vec{r} \cdot \vec{u}_r &= a\vec{u}_r \cdot \vec{u}_r + b\vec{u}_\theta \cdot \vec{u}_r \\
&= a(1) + b(0) \\
&= a \\
\vec{r} \cdot \vec{u}_\theta &= a\vec{u}_r \cdot \vec{u}_\theta + b\vec{u}_\theta \cdot \vec{u}_\theta \\
&= a(0) + b(1) \\
&= b
\end{aligned}$$

This means that if we are given two expressions in terms of $\vec{u}_r$ and $\vec{u}_\theta$ that are equal to each other, then we can set the r and θ components equal to each other.

Newton's Law in Polar Coordinates

Now we are ready to start using Newton's Law. We will use polar coordinates, where r and θ are functions of time t.

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$$\begin{aligned}
\vec{r} &= (r\cos(\theta), r\sin(\theta)) \\
&= r\vec{u}_r \\
\vec{v} &= \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt}
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&= \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{d\theta}\frac{d\theta}{dt} \\
&= \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta
\end{aligned}$$

Perihelion

At perihelion, where r is at a minimum,

$$\vec{v}\big|_{t=t_0} = \left.\frac{dr}{dt}\right|_{t=t_0} \vec{u}_r\big|_{\theta=0} + r_0 \left.\frac{d\theta}{dt}\right|_{t=t_0} \vec{u}_\theta\big|_{\theta=0}$$

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If we let s_0 be the speed at perihelion, then we have:

$$s_0 = r_0 \left.\frac{d\theta}{dt}\right|_{t=t_0} \tag{2}$$

Central Acceleration

$$\vec{v} = \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta$$

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$$\vec{a} = \frac{d^2 r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\vec{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta + r\frac{d\theta}{dt}\frac{d\vec{u}_\theta}{dt}$$

Central Acceleration

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&= \frac{d^2r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\vec{u}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta + r\frac{d\theta}{dt}\frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt}
\end{aligned}
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&= \frac{d^2 r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\vec{u}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta + r\frac{d\theta}{dt}\frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} \\
&= \frac{d^2 r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\vec{u}_r
\end{aligned}$$

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&= \frac{d^2r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\vec{u}_r}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta + r\frac{d\theta}{dt}\frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} \\
&= \frac{d^2r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\vec{u}_r \\
&= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\vec{u}_\theta
\end{aligned}$$

Inverse Square Law

We have another expression for $\vec{a}$ from Newton's Law of Gravity: $\vec{a}$ points from the planet to the Sun (in the direction of $-\vec{u}_r$), and has magnitude GM/r^2.

$$\vec{a} = -\frac{GM}{r^2}\vec{u}_r$$

By setting equal the r and θ components of $\vec{a}$, we end up with the following two equations:

$$\begin{gathered}\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \\ 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0\end{gathered}$$

Integrating Factors

- We are now faced with some fairly complicated differential equations. We do have the advantage that we know what the answer is; we will keep in mind the polar equation for an ellipse and act accordingly.

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- We are now faced with some fairly complicated differential equations. We do have the advantage that we know what the answer is; we will keep in mind the polar equation for an ellipse and act accordingly.
- Another technique that we will use repeatedly is that of integrating factors. Note:

$$\begin{aligned}\frac{d}{dx}\left(ye^{F(x)}\right) &= \frac{dy}{dx}e^{F(x)} + yF'(x)e^{F(x)} \\ &= \left(\frac{dy}{dx} + F'(x)y\right)e^{F(x)}\end{aligned}$$

Integrating Factors (continued)

- We will frequently encounter an expression of the form

  $$\frac{dy}{dx} + f(x)y$$

  and wish that we could integrate it. If we can find an antiderivative $F(x)$ for $f(x)$, then multiplying the expression by $e^{F(x)}$ will turn it into the derivative of $ye^{F(x)}$, which will permit us to go ahead and integrate.

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- In words: given the sum of a derivative of y and a function times y, integrate the function, exponentiate it, and multiply through. You will then have an expression that you can integrate.

Eliminating Time from the System

We currently have three variables (r, θ, and t) and we ultimately only want to see r and θ. We need to eliminate time from our system. We will work with the θ component of acceleration to do so.

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0$$

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$$\begin{aligned} 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0 \\ \frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = 0 \end{aligned}$$

Eliminating Time from the System (continued)

We will create an integrating factor, where $d\theta/dt$ is playing the role of y:

$$\int \frac{2}{r}\frac{dr}{dt}\,dt = \int \frac{2dr}{r}$$

Eliminating Time from the System (continued)

We will create an integrating factor, where $d\theta/dt$ is playing the role of y:

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e^{2\ln(r)} &= r^2 \\
r^2\left(\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\right) &= 0
\end{aligned}$$

Eliminating Time from the System (continued)

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e^{2\ln(r)} &= r^2 \\
r^2\left(\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\right) &= 0 \\
r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} &= 0
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r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} &= 0 \\
\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= 0
\end{aligned}$$

Eliminating Time from the System (continued)

We will create an integrating factor, where $d\theta/dt$ is playing the role of y:

$$\begin{aligned}
\int \frac{2}{r}\frac{dr}{dt}\,dt &= \int \frac{2dr}{r} \\
&= 2\ln(r) \\
e^{2\ln(r)} &= r^2 \\
r^2\left(\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\right) &= 0 \\
r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} &= 0 \\
\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= 0 \\
\int_{t_0}^{t} \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\,dt &= \int_{t_0}^{t} 0\,dt
\end{aligned}$$

Eliminating Time from the System (continued)

$$r^2 \frac{d\theta}{dt} - r_0^2 \left. \frac{d\theta}{dt} \right|_{t=t_0} = 0$$

Eliminating Time from the System (continued)

$$\begin{aligned} r^2\frac{d\theta}{dt} - r_0^2\ \frac{d\theta}{dt}\bigg|_{t=t_0} &= 0 \\ r^2\frac{d\theta}{dt} &= r_0^2\ \frac{d\theta}{dt}\bigg|_{t=t_0} \end{aligned}$$

Eliminating Time from the System (continued)

$$\begin{aligned} r^2 \frac{d\theta}{dt} - r_0^2 \left.\frac{d\theta}{dt}\right|_{t=t_0} &= 0 \\ r^2 \frac{d\theta}{dt} &= r_0^2 \left.\frac{d\theta}{dt}\right|_{t=t_0} \\ &= r_0 s_0 \qquad \text{(See Equation \#2)} \end{aligned}$$

Eliminating Time from the System (continued)

$$\begin{aligned} r^2 \frac{d\theta}{dt} - r_0^2 \left. \frac{d\theta}{dt} \right|_{t=t_0} &= 0 \\ r^2 \frac{d\theta}{dt} &= r_0^2 \left. \frac{d\theta}{dt} \right|_{t=t_0} \\ &= r_0 s_0 \qquad \text{(See Equation \#2)} \\ \frac{d\theta}{dt} &= \frac{r_0 s_0}{r^2} \end{aligned}$$

Eliminating Time from the System (continued)

Since the right-hand side of the last equation does not contain t, we can use the chain rule now to eliminate t from our system.

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

Eliminating Time from the System (continued)

Since the right-hand side of the last equation does not contain t, we can use the chain rule now to eliminate t from our system.

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Since the right-hand side of the last equation does not contain t, we can use the chain rule now to eliminate t from our system.

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Since the right-hand side of the last equation does not contain t, we can use the chain rule now to eliminate t from our system.

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Eliminating Time from the System (continued)

$$\frac{d^2 r}{dt^2} = -2 r_0 s_0 r^{-3} \frac{dr}{dt} \frac{dr}{d\theta} + r_0 s_0 r^{-2} \frac{d}{dt} \frac{dr}{d\theta}$$

Eliminating Time from the System (continued)

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&= -2 r_0 s_0 r^{-3} \frac{dr}{d\theta} \frac{d\theta}{dt} \frac{dr}{d\theta} + r_0 s_0 r^{-2} \frac{d^2 r}{d\theta^2} \frac{d\theta}{dt}
\end{aligned}$$

Eliminating Time from the System (continued)

$$\begin{aligned}
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&= -2 r_0^2 s_0^2 r^{-5} \left(\frac{dr}{d\theta} \right)^2 + r_0^2 s_0^2 r^{-4} \frac{d^2 r}{d\theta^2}
\end{aligned}$$

Eliminating Time from the System (continued)

Now we can take the r component of acceleration and express it entirely in terms of r and θ:

$$\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = -\frac{GM}{r^2}$$

Eliminating Time from the System (continued)

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\end{aligned}$$

Feeling better yet?

Kepler's Second Law

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- At least we have proved Kepler's Second Law: the line joining a planet to the Sun sweeps out equal areas in equal times as the planet travels around its orbit.
- In polar coordinates, the area swept out is found by integrating $r^2/2$:

$$\begin{aligned} A &= \int_{\theta=\theta_0}^{\theta=\theta} \frac{r^2}{2}\, d\theta \\ &= \int_{t=t_0}^{t=t} \frac{r^2}{2}\frac{d\theta}{dt}\, dt \end{aligned}$$

  But the integrand of the last integral is a constant, namely $r_0 s_0/2$. Therefore the areas swept out over equal times are indeed equal.

Simplification: Sinusoid

- Why is the *r* component of acceleration so ugly?

Simplification: Sinusoid

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- Partly it's because the polar equation for an ellipse has the cosine in the denominator of a fraction. We would be better off working with the reciprocal of r, which would then be a multiple of a cosine plus a constant. Such expressions are called sinusoids.

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- Partly it's because the polar equation for an ellipse has the cosine in the denominator of a fraction. We would be better off working with the reciprocal of r, which would then be a multiple of a cosine plus a constant. Such expressions are called sinusoids.
- Let $u = 1/r$. We will express the r component of acceleration in terms of u, and it will become much easier to work with.

Simplification: Sinusoid (continued)

$$u = r^{-1}$$

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Simplification: Sinusoid (continued)

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u &= r^{-1} \\
u_0 &= r_0^{-1} \\
r &= u^{-1} \\
\frac{dr}{d\theta} &= -u^{-2}\frac{du}{d\theta} \\
\left.\frac{dr}{d\theta}\right|_{t=t_0} &= 0
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\left.\frac{dr}{d\theta}\right|_{t=t_0} &= 0 \\
\left.\frac{du}{d\theta}\right|_{t=t_0} &= 0 \\
\frac{d^2 r}{d\theta^2} &= 2u^{-3}\left(\frac{du}{d\theta}\right)^2 - u^{-2}\frac{d^2 u}{d\theta^2}
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Simplification: Sinusoid (continued)

Making these substitutions, the *r* component of acceleration becomes:

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Much, much better.

Simplification: Cosine

We can do more. If u is a sinusoid, it is a constant added to a multiple of a cosine. From the last equation, the constant looks to be $GM/r_0^2 s_0^2$. Let $u = v + GM/r_0^2 s_0^2$. All derivatives of v are the same as the derivatives of u, since u and v only differ by a constant.

$$u_0 = r_0^{-1}$$

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&= \frac{r_0 s_0^2 - GM}{r_0^2 s_0^2}
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Simplification: Cosine (continued)

$$\left.\frac{dv}{d\theta}\right|_{t=t_0} = \left.\frac{du}{d\theta}\right|_{t=t_0}$$

Simplification: Cosine (continued)

$$\begin{aligned} \left.\frac{dv}{d\theta}\right|_{t=t_0} &= \left.\frac{du}{d\theta}\right|_{t=t_0} \\ &= 0 \end{aligned}$$

Simplification: Cosine (continued)

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\frac{d^2 u}{d\theta^2} + u &= \frac{GM}{r_0^2 s_0^2}
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\frac{d^2v}{d\theta^2} + v + \frac{GM}{r_0^2 s_0^2} &= \frac{GM}{r_0^2 s_0^2}
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\frac{d^2v}{d\theta^2} + v &= 0
\end{aligned}$$

Solution

We believe that v should be a multiple of a cosine. Let $v = \cos(\theta) w$; then w should eventually be a constant and the equation should be really simple.

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\frac{dv}{d\theta} &= -\sin(\theta)w + \cos(\theta)\frac{dw}{d\theta}
\end{aligned}$$

Solution (continued)

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&= 0 \\
\frac{d^2v}{d\theta^2} &= -\cos(\theta)w - \sin(\theta)\frac{dw}{d\theta} - \sin(\theta)\frac{dw}{d\theta} + \cos(\theta)\frac{d^2w}{d\theta^2}
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&= -\cos(\theta)w - 2\sin(\theta)\frac{dw}{d\theta} + \cos(\theta)\frac{d^2w}{d\theta^2}
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-2\sin(\theta)\frac{dw}{d\theta} + \cos(\theta)\frac{d^2 w}{d\theta^2} &= 0 \\
\frac{d^2 w}{d\theta^2} - 2\tan\theta\frac{dw}{d\theta} &= 0
\end{aligned}$$

Solution (continued)

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\frac{d^2w}{d\theta^2} - 2\tan\theta\frac{dw}{d\theta} &= 0\\
\cos^2(\theta)\frac{d^2w}{d\theta^2} - 2\sin(\theta)\cos(\theta)\frac{dw}{d\theta} &= 0\\
\frac{d}{d\theta}\left(\cos^2(\theta)\frac{dw}{d\theta}\right) &= 0
\end{aligned}$$

Solution (continued)

$$\int_{\theta_0}^{\theta} \frac{d}{d\theta}\left(\cos^2(\theta)\frac{dw}{d\theta}\right)\, d\theta = \int_{\theta_0}^{\theta} 0 \, d\theta$$

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\cos^2(\theta)\frac{dw}{d\theta} &= 0 \\
\frac{dw}{d\theta} &= 0 \\
\int_0^{\theta} \frac{dw}{d\theta}\, d\theta &= \int_0^{\theta} 0\, d\theta
\end{aligned}
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Solution (continued)

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$$\begin{aligned} w - w_0 &= 0 \\ w &= w_0 \\ &= \frac{r_0 s_0^2 - GM}{r_0^2 s_0^2} \end{aligned}$$

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Solution (continued)

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r &= \frac{1}{u} \\
&= \frac{r_0^2 s_0^2}{\left(r_0 s_0^2 - GM\right)\cos(\theta) + GM} \\
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which is the equation for an ellipse, proving Kepler's First Law.

Solution (continued)

Before we can proceed to Kepler's Third Law, we need to work with a and e. The numerator is $a\left(1-e^2\right)$ and the denominator is $1+e\cos(\theta)$.

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Solution (continued)

Before we can proceed to Kepler's Third Law, we need to work with a and e. The numerator is $a\left(1-e^2\right)$ and the denominator is $1+e\cos(\theta)$.

$$\begin{aligned} e &= \frac{r_0 s_0^2}{GM} - 1 \\ a\left(1-e^2\right) &= \frac{r_0^2 s_0^2}{GM} \end{aligned}$$

Period

To find the period of the orbit T, remember that the area of an ellipse is πab.

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Kepler's Third Law claims that a^3/T^2 is a constant for all planets, depending only on the mass of the Sun.

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&= \frac{a r_0^2 s_0^2}{4\pi^2 b^2} \\
&= \frac{a r_0^2 s_0^2}{4\pi^2 \left(a^2 - c^2\right)} \\
&= \frac{a r_0^2 s_0^2}{4\pi^2 a^2 \left(1 - e^2\right)} \\
&= \frac{r_0^2 s_0^2}{4\pi^2 a \left(1 - e^2\right)}
\end{aligned}$$

Kepler's Third Law (continued)

$$= \frac{r_0^2 s_0^2}{4\pi^2 \left(r_0^2 s_0^2 / GM \right)}$$

Kepler's Third Law (continued)

$$
\begin{aligned}
&= \frac{r_0^2 s_0^2}{4\pi^2 \left(r_0^2 s_0^2 / GM \right)} \\
&= \frac{GM}{4\pi^2}
\end{aligned}
$$

Kepler's Third Law (continued)

$$
\begin{aligned}
&= \frac{r_0^2 s_0^2}{4\pi^2 \left(r_0^2 s_0^2 / GM \right)} \\
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which is a constant depending only on the mass of the Sun.