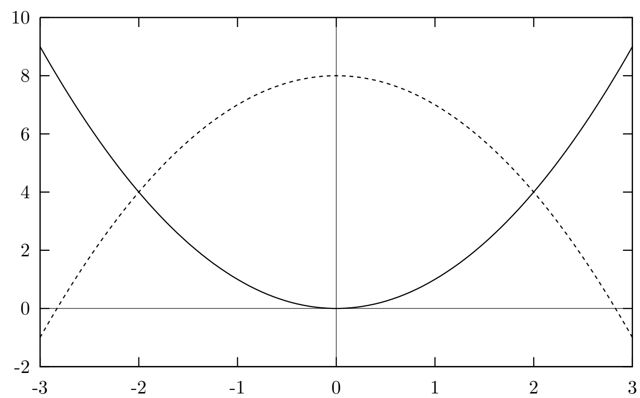
Exam #3 Solutions

Math 321-A

Thursday, May 8, 2008

1. The two vectors are $\vec{u}_r = \langle \cos(\theta), \sin(\theta) \rangle$ and $\vec{u}_\theta = \langle -\sin(\theta), \cos(\theta) \rangle$. $\vec{u}_r$ is a unit vector in the direction from the origin to the planet; $\vec{u}_\theta$ is obtained by rotating $\vec{u}_r$ counter-clockwise by a right angle.

2. Here is the region bounded by $y = 8 - x^2$ and $y = x^2$.



The lower curve is $y = x^2$ and the upper curve is $y = 8 - x^2$. The curves intersect at $x^2 = 8 - x^2$, $2x^2 = 8$, $x^2 = 4$, $x = \pm 2$. The region is described by $x^2 \le y \le 8 - x^2$, $-2 \le x \le 2$.

We will find the surface area by integrating $\sqrt{f_x^2 + f_y^2 + 1}$.

$$\begin{aligned} f(x,y) &= e^{x^2} + \cos(xy) \\ f_x &= 2xe^{x^2} - y\sin(xy) \\ f_y &= -x\sin(xy) \\ f_x^2 + f_y^2 + 1 &= 4x^2e^{2x^2} - 4xye^{x^2}\sin(xy) + y^2\sin^2(xy) + x^2\sin^2(xy) + 1 \\ \text{surface area } &= \int_{-2}^{2}\int_{x^2}^{8-x^2} \sqrt{4x^2e^{2x^2} - 4xye^{x^2}\sin(xy) + y^2\sin^2(xy) + x^2\sin^2(xy) + 1}\,dy\,dx \end{aligned}$$

3. The density function is positive only in the first quadrant. The piece of the first quadrant that we are interested in is $0 \le y \le 4 - x$, $0 \le x \le 4$.

$$\begin{aligned} \Pr(y \le 4 - x) &= \int_0^4 \int_0^{4-x} 15e^{-3x-5y}\,dy\,dx \\ &= \int_0^4 -3e^{-3x-5y}\Big|_{y=0}^{y=4-x}\,dx \\ &= \int_0^4 -3\left(e^{2x-20} - e^{-3x}\right)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \left(-\frac{3e^{2x-20}}{2} - e^{-3x}\right)\Bigg|_{x=0}^{x=4} \\
&= \left(-\frac{3e^{-12}}{2} - e^{-12}\right) - \left(-\frac{3e^{-20}}{2} - 1\right) \\
&= -\frac{5e^{-12}}{2} + \frac{3e^{-20}}{2} + 1 \\
&= 0.999985
\end{aligned}
$$

4. The surface $z = 9 - x^2 - y^2$ intersects the xy-plane at $z = 0$: $0 = 9 - x^2 - y^2$, i.e., $x^2 + y^2 = 9$, which is a circle centered at the origin of radius 3.

   The region is described by $0 \le z \le 9 - r^2$, $0 \le r \le 3$, and $0 \le \theta \le 2\pi$.

$$
\begin{aligned}
V &= \iiint_R 1\, dV \\
&= \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^3 z\Big|_{z=0}^{z=9-r^2} dr\, d\theta \\
&= \int_0^{2\pi} \int_0^3 \left(9 - r^2\right) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^3 \left(9r - r^3\right) dr\, d\theta \\
&= \int_0^{2\pi} \left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\Bigg|_{r=0}^{r=3} d\theta \\
&= \int_0^{2\pi} \left(\frac{81}{2} - \frac{81}{4}\right) d\theta \\
&= \int_0^{2\pi} \frac{81}{4}\, d\theta \\
&= \frac{81}{4}(2\pi) \\
&= \frac{81\pi}{2} \\
&= 127.235
\end{aligned}
$$

5. The partition for x is $1, 2, 3$ and the partition for y is $4, 6, 8$. $\Delta x = 1$ and $\Delta y = 2$. The midpoints for x are 1.5 and 2.5, and the midpoints for y are 5 and 7.

$$
\begin{aligned}
\sum_{i=1}^{2}\sum_{j=1}^{2} x_i^2 y_j \Delta x \Delta y &= (1.5)^2(5)(1)(2) + (1.5)^2(7)(1)(2) + (2.5)^2(5)(1)(2) + (2.5)^2(7)(1)(2) \\
&= 204
\end{aligned}
$$

6. The first derivative test does not work for functions of two variables, since there are an infinite number of directions to check.

   The second partials test involves checking three different second derivatives as opposed to just one.

   We find critical points by working with the gradient for functions of more than one variable.

7. We know that the unit circle is closed and bounded, so it has absolute extrema.

$$
\begin{aligned}
f(x, y) &= xy \\
\nabla f &= \langle y, x \rangle \\
g(x, y) &= x^2 + y^2 - 1 \\
\nabla g &= \langle 2x, 2y \rangle \\
y &= 2\lambda x \\
x &= 2\lambda y \\
x^2 + y^2 &= 1
\end{aligned}
$$

Note that x and y can't both be 0 by the previous equation. Then neither can be 0 by the two equations before that.

$$
\begin{aligned}
x &= 2\lambda(2\lambda x) \\
4\lambda^2 x - x &= 0 \\
(4\lambda^2 - 1)x &= 0 \\
4\lambda^2 - 1 &= 0 \\
4\lambda^2 &= 1 \\
\lambda^2 &= \frac{1}{4} \\
\lambda &= \pm\frac{1}{2}
\end{aligned}
$$

If $\lambda = -1/2$:

$$
\begin{aligned}
y &= -x \\
x^2 + x^2 &= 1 \\
2x^2 &= 1 \\
x &= \pm\frac{1}{\sqrt{2}} \\
(x, y) &= \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \\
f(x, y) &= -\frac{1}{2}
\end{aligned}
$$

If $\lambda = 1/2$:

$$
\begin{aligned}
y &= x \\
x^2 + x^2 &= 1 \\
x &= \pm\frac{1}{\sqrt{2}} \\
(x, y) &= \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\
f(x, y) &= \frac{1}{2}
\end{aligned}
$$

The absolute minima are at $\pm(1/\sqrt{2}, -1/\sqrt{2})$ and the absolute maxima are at $\pm(1/\sqrt{2}, 1/\sqrt{2})$.

8. By symmetry all of the coordinates should be the same.

   Since the density is constant, assume it is 1. Then the mass of the region is one eighth of the volume of a sphere of radius 1, $(1/8)4\pi(1)^3/3 = \pi/6$.

   The region can be described by $0 \le \rho \le 1$, $0\phi \le \pi/2$, and $0 \le \theta \le \pi/2$.

$$\begin{aligned}
\iiint z\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\cos(\phi)\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3\cos(\phi)\sin(\phi)\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \left.\frac{\rho^4\cos(\phi)\sin(\phi)}{4}\right|_{\rho=0}^{\rho=1} d\phi\,d\theta \\
&= \frac{1}{4}\int_0^{\pi/2}\int_0^{\pi/2}\cos(\phi)\sin(\phi)\,d\phi\,d\theta
\end{aligned}$$

   Let $u = \sin(\phi)$, $du = \cos(\phi)\,d\phi$, $0 \le u \le 1$.

$$\begin{aligned}
&= \frac{1}{4}\int_0^{\pi/2}\int_0^1 u\,du\,d\theta \\
&= \frac{1}{4}\int_0^{\pi/2}\left.\frac{u^2}{2}\right|_{u=0}^{u=1} d\theta \\
&= \frac{1}{8}\int_0^{\pi/2} d\theta \\
&= \frac{1}{8}\frac{\pi}{2} \\
&= \frac{\pi}{16} \\
\bar{z} &= \frac{\pi/16}{\pi/6} \\
&= \frac{3}{8} \\
&= 0.375
\end{aligned}$$

   and $(\bar{x}, \bar{y}, \bar{z}) = (0.375, 0.375, 0.375)$.

9. We will solve for the critical point(s) by setting the gradient equal to $\vec{0}$, since the gradient is defined everywhere.

$$\begin{aligned}
f(x,y) &= xe^x - ye^y \\
\nabla f &= \langle e^x + xe^x, -e^y - ye^y\rangle \\
&= \langle (x+1)e^x, -(y+1)e^y\rangle \\
&= \langle 0, 0\rangle \\
x &= -1 \\
y &= -1 \\
f_{xx} &= e^x + (x+1)e^x \\
&= (x+2)e^x \\
f_{yy} &= -(y+2)e^y \\
f_{xy} &= 0 \\
D &= f_{xx}(-1,-1)f_{yy}(-1,-1) - f_{xy}(-1,-1)^2
\end{aligned}$$

$$= e^{-1}\left(-e^{-1}\right) - 0^2$$
$$= -e^{-2}$$

and the only critical point is at $(-1, -1)$, which is a saddle point.

10. The example that we did in class was $f(x) = x$ for $0 < x < 1$. A simple variation would be $f(x) = x + 1$ for $0 < x < 1$. It takes smaller and smaller values as $x \to 0$, never quite reaching $f(x) = 1$; similarly it takes larger and larger values as $x \to 1$, never quite reaching $f(x) = 2$.