Final Exam Solutions

Math 321-A

Thursday, May 14, 2009

1.

$$
\begin{aligned}
\cos(\theta) &= \frac{\langle -2, 6, -5\rangle \cdot \langle 7, -2, 7\rangle}{|\langle -2, 6, -5\rangle|\,|\langle 7, -2, 7\rangle|} \\
&= \frac{-14 - 12 - 35}{\sqrt{4 + 36 + 25}\,\sqrt{49 + 4 + 49}} \\
&= -\frac{61}{\sqrt{65}\,\sqrt{102}} \\
\theta &= \cos^{-1}\left(-\frac{61}{\sqrt{65}\,\sqrt{102}}\right) \\
&= 2.418
\end{aligned}
$$

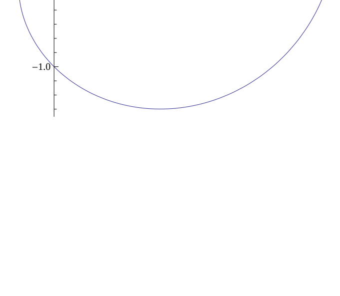
2. We can draw curves along the surface $z = f(x, y)$ corresponding to fixed values of x and to fixed values of y. (*Mathematica* does this by default.) When y is held fixed, $\partial z/\partial x$ is the slope of the curve for each x. When x is held fixed, $\partial z/\partial y$ is the slope of the curve for each y.

3. The normal component of acceleration measures the turning of the curve; if it is always zero, the curve never turns but must proceed in a straight line.

4. The gradient of $f(x, y, z) = xy^2z^3$ at the point P will be the normal vector for the tangent plane and the direction vector for the normal line.

$$
\begin{aligned}
f(x, y, z) &= xy^2z^3 \\
\nabla f &= \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle \\
\nabla f(1, 2, 3) &= \langle 108, 108, 108\rangle \\
\text{Tangent Plane: }\ 108(x-1) + 108(y-2) + 108(z-3) &= 0 \\
\text{Normal Line: }\ (x, y, z) &= (1 + 108t, 2 + 108t, 3 + 108t)
\end{aligned}
$$

5. Here is a picture of the graph:

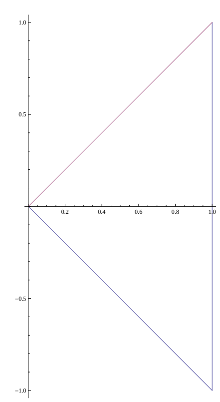


$$\begin{aligned}
x &= r\cos(\theta) \\
&= (1+\cos(\theta))\cos(\theta) \\
\frac{dx}{d\theta} &= -\sin(\theta)\cos(\theta) + (1+\cos(\theta))(-\sin(\theta)) \\
&= -\sin(\theta) - 2\sin(\theta)\cos(\theta) \\
y &= r\sin(\theta) \\
&= (1+\cos(\theta))\sin(\theta) \\
\frac{dy}{d\theta} &= -\sin(\theta)\sin(\theta) + (1+\cos(\theta))\cos(\theta) \\
&= -\sin^2(\theta) + \cos(\theta) + \cos^2(\theta) \\
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\
&= \frac{-\sin^2(\theta) + \cos(\theta) + \cos^2(\theta)}{-\sin(\theta) - 2\sin(\theta)\cos(\theta)}
\end{aligned}$$

This is undefined when the denominator is zero.

$$\begin{aligned}
-\sin(\theta) - 2\sin(\theta)\cos(\theta) &= 0 \\
&= -\sin(\theta)(1+2\cos(\theta)) \\
\text{If } \sin(\theta) = 0\text{:}\ \ \theta &= n\pi \\
(x,y) &= (2,0) \text{ or } (0,0) \\
\text{If } \cos(\theta) = -\frac{1}{2}\text{:}\ \ \theta &= \pm\frac{\pi}{3} + 2n\pi \\
(x,y) &= \left(-\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right)
\end{aligned}$$

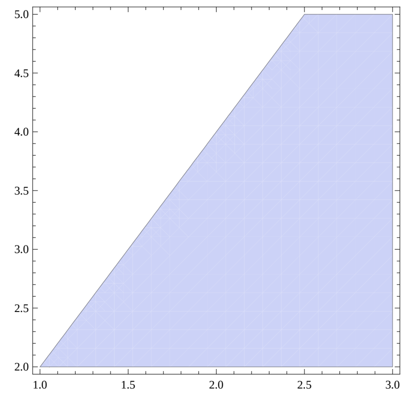
6. Here is a picture of the graph:



$$\begin{aligned}
\text{Area} &= \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\,dr \\
&= \int_{-\pi/4}^{\pi/4} \frac{\sec^2(\theta)}{2}\,dr \\
&= \left.\frac{\tan(\theta)}{2}\right|_{-\pi/4}^{\pi/4}
\end{aligned}$$

$$= \frac{1}{2} - \left(-\frac{1}{2}\right)$$
$$= 1$$

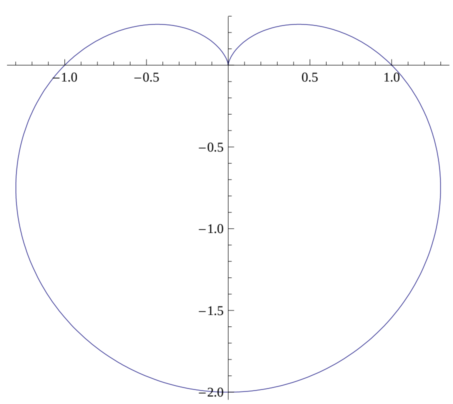
7. Note that $y/x \leq 2$ for positive x and y means that $y \leq 2x$.



If we integrate from top to bottom, we will need to do it over two regions, since the top boundary goes from slanted to horizontal at $x = 5/2$. We will integrate from left to right, with $y/2 \leq x \leq 3$ and $2 \leq y \leq 5$.

$$\begin{aligned}
\Pr(Y/X \leq 2) &= \int_2^5 \int_{y/2}^3 \frac{x+y}{33}\, dx\, dy \\
&= \int_2^5 \left.\frac{x^2/2 + xy}{33}\right|_{x=y/2}^{x=3} dy \\
&= \int_2^5 \left(\frac{9/2 + 3y}{33} - \frac{y^2/8 + y^2/2}{33}\right) dy \\
&= \int_2^5 \frac{-5y^2 + 24y + 36}{264}\, dy \\
&= \left.\frac{-5y^3/3 + 12y^2 + 36y}{264}\right|_{y=2}^{y=5} \\
&= \frac{-625/3 + 300 + 180}{264} - \frac{-40/3 + 48 + 72}{264} \\
&= \frac{5}{8} \\
&= 62.5\%
\end{aligned}$$

8. Here is a graph of the curve for $0 \leq \theta \leq 2\pi$.



$$\begin{aligned}
\frac{ds}{dt} &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \\
&= \sqrt{(1-\sin(\theta))^2 + (-\cos(\theta))^2} \\
&= \sqrt{2 - 2\sin(\theta)} \\
\text{arc-length} &= \int_0^{2\pi} \sqrt{2-2\sin(\theta)}\, d\theta \\
&= 8
\end{aligned}$$

9. Since $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$ for any two vectors, if we let $\vec{w} = \vec{v}$, we have that $\vec{v} \times \vec{v} = -\vec{v} \times \vec{v}$, or $2\vec{v} \times \vec{v} = \vec{0}$. Dividing out by 2 yields $\vec{v} \times \vec{v} = \vec{0}$.

10.

$$\begin{aligned}
\overrightarrow{T} \cdot \left(\overrightarrow{N} \times \overrightarrow{B}\right) &= \overrightarrow{T} \cdot \overrightarrow{T} \\
&= 1
\end{aligned}$$

since $\overrightarrow{N} \times \overrightarrow{B} = \overrightarrow{T}$ and $\overrightarrow{T}$ has length 1.

11. We start by connecting the point Q to the line with a vector. The easiest point to find on the line is the point $P = (4, -5, -9)$ when $t = 0$. $\overrightarrow{PQ} = Q - P = \langle 5, 3, 11 \rangle$.

We will next split $\overrightarrow{PQ}$ into components that are parallel and perpendicular to the line. (We will want the perpendicular component.) The parallel component is the projection of $\overrightarrow{PQ}$ onto $\overrightarrow{A} = \langle -6, -5, -6 \rangle$, the direction vector for the line.

$$\begin{aligned}
\text{proj}_{\overrightarrow{A}}\left(\overrightarrow{PQ}\right) &= \frac{\overrightarrow{PQ} \cdot \overrightarrow{A}}{\overrightarrow{A} \cdot \overrightarrow{A}} \overrightarrow{A} \\
&= \frac{-30 - 15 - 66}{36 + 25 + 36} \langle -6, -5, -6 \rangle \\
&= -\frac{111}{97} \langle -6, -5, -6 \rangle \\
&= \left\langle \frac{666}{97}, \frac{555}{97}, \frac{666}{97} \right\rangle
\end{aligned}$$

The perpendicular component is what remains, $\overrightarrow{PQ} - \text{proj}_{\overrightarrow{A}}\left(\overrightarrow{PQ}\right) = \langle -\frac{181}{97}, -\frac{264}{97}, \frac{401}{97} \rangle$.

The perpendicular distance is the length of the perpendicular component of $\overrightarrow{PQ}$.

$$\begin{aligned}
\left\| \left\langle -\frac{181}{97}, -\frac{264}{97}, \frac{401}{97} \right\rangle \right\| &= \frac{1}{97} \sqrt{181^2 + 264^2 + 401^2} \\
&= \frac{\sqrt{263{,}258}}{97} \\
&= 5.290
\end{aligned}$$

12. We start by finding the gradient.

$$\begin{aligned} f(x,y) &= x^2y + 2xy^2 + y \\ \nabla f &= \left\langle 2xy + 2y^2, x^2 + 4xy + 1 \right\rangle \end{aligned}$$

Since the gradient is always defined, we solve for critical points by setting the gradient equal to zero.

$$\begin{aligned} 2xy + 2y^2 &= 0 \\ 2y(x+y) &= 0 \\ y &= 0 \text{ or} \\ y &= -x \\ \text{If } y = 0\text{: } x^2 + 4xy + 1 &= x^2 + 1 \\ &= 0 \end{aligned}$$

which has no solution.

$$\begin{aligned} \text{If } y = -x\text{: } x^2 + 4xy + 1 &= -3x^2 + 1 \\ &= 0 \\ x &= \pm\frac{1}{\sqrt{3}} \\ y &= \mp\frac{1}{\sqrt{3}} \end{aligned}$$

and there are two critical points.

$$\begin{aligned} f_{xx} &= 2y \\ f_{xy} &= 2x + 4y \\ f_{yy} &= 4x \end{aligned}$$

Critical Point	f_{xx}	f_{yy}	f_{xy}	D	Conclusion
$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$-\frac{2}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}}$	-4	Saddle Point
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{2}{\sqrt{3}}$	$-\frac{4}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	-4	Saddle Point

13.

Sub-rectangle	(x^*, y^*)	$f(x^*, y^*)$	Δx	Δy	$f(x^*,y^*)\Delta x\Delta y$
$[-1,0]\times[3,6]$	$(-0.5, 4.5)$	1.125	1	3	3.375
$[-1,0]\times[6,9]$	$(-0.5, 7.5)$	1.875	1	3	5.625
$[0,1]\times[3,6]$	$(0.5, 4.5)$	1.125	1	3	3.375
$[0,1]\times[6,9]$	$(0.5, 7.5)$	1.875	1	3	5.625
					18.000

and the Riemann Sum is 18.

14. When the plane slices across the cone parallel to the one of the lines of the cone through the vertex, the section is a parabola. When the plane angles more toward the axis the section is an ellipse. When the plan angles away from the axis striking the other half of the cone the section is a hyperbola.

15. The region can be described $0 \le r \le 4$ and $0 \le \theta \le \pi/2$. The integrand is $\sqrt{x^2+y^2} = r$, and the differential is $dA = r\,dr\,d\theta$.

$$\begin{aligned}
\iint\limits_{\text{region}} \sqrt{x^2+y^2}\,dA &= \int_0^{\pi/2}\int_0^4 (r)r\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^4 r^2\,dr\,d\theta \\
&= \int_0^{\pi/2} \left.\frac{r^3}{3}\right|_{r=0}^{r=4} d\theta \\
&= \int_0^{\pi/2} \frac{64}{3}\,d\theta \\
&= \frac{64}{3}\frac{\pi}{2} \\
&= \frac{32\pi}{3} \\
&= 33.510
\end{aligned}$$

16. The area of the triangle is half the length of $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\begin{aligned}
\overrightarrow{PQ} &= Q - P \\
&= \langle 15, 1 - 2\rangle \\
\overrightarrow{PR} &= R - P \\
&= \langle 16, 10, -2\rangle \\
\overrightarrow{PQ} \times \overrightarrow{PR} &= \langle 18, -2, 134\rangle \\
\text{triangle area} &= \frac{\sqrt{18^2 + (-2)^2 + 134^2}}{2} \\
&= 67.609
\end{aligned}$$

17. We will substitute the equation for the line into the equation for the plane.

$$\begin{aligned}
9x - 2y - 9z &= -5 \\
9(4t-5) - 2(4t-8) - 9(9t+3) &= -5 \\
-53t &= 51 \\
t &= -\frac{51}{53} \\
(x,y,z) &= \left(-\frac{469}{53}, -\frac{628}{53}, -\frac{300}{53}\right) \\
&= (-8.849, -11.849, -5.660)
\end{aligned}$$

18. The surface is a hyperboloid of one sheet.

19.

$$\vec{v} = \langle 4, 6, 8\rangle$$

$$
\begin{aligned}
\vec{a} &= \langle 9, 2, 7 \rangle \\
\vec{v} \times \vec{a} &= \langle 26, 44, -46 \rangle \\
\left|\vec{v} \times \vec{a}\right| &= \sqrt{26^2 + 44^2 + (-46)^2} \\
&= \sqrt{4728} \\
&= 2\sqrt{1182} \\
\left|\vec{v}\right| &= \sqrt{4^2 + 6^2 + 8^2} \\
&= \sqrt{116} \\
&= 2\sqrt{29} \\
\kappa &= \frac{\left|\vec{v} \times \vec{a}\right|}{\left|\vec{v}\right|^3} \\
&= \frac{2\sqrt{1182}}{232\sqrt{29}} \\
&= 0.055
\end{aligned}
$$

20. There is a plane that goes through that point that is parallel to the unit tangent and unit normal vectors. If the torsion is positive, the curve moves up out of the plane as it bends counterclockwise.